THE WIENER CLOSURE THEOREMS FOR ABSTRACT WIENER SPACES

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Abstract. We introduce $\mathcal{L}_1$ and $\mathcal{L}_2$ translates for functions in $\mathcal{L}_1(\mu)$ and $\mathcal{L}_2(\mu)$ where $\mu$ is a Gaussian measure on a Banach space. With these translates and the Fourier-Wiener transforms defined by Cameron and Martin we obtain Wiener's closure theorem in $\mathcal{L}_2(\mu)$ and in $\mathcal{L}_1(\mu)$. Using the $\mathcal{L}_1(\mu)$ results we indicate the analogue of the Wiener-Pitt Tauberian theorems for this setup.

1. Introduction. Let $\mu$ be the Wiener measure on $C[0, 1]$ and $\mathcal{L}_2(\mu)$ be the space of square integrable Borel functions with respect to $\mu$. For $f \in \mathcal{L}_2(\mu)$, the Fourier-Wiener transform was defined by Cameron and Martin [3]. In this paper we extend this notion to abstract Wiener spaces [6] and obtain an analogue of Wiener’s closure theorem [10] for $\mathcal{L}_2(\mu)$. Our main effort however is to obtain an analogue of Wiener’s closure theorem for $\mathcal{L}_1(\mu)$. From this theorem one can easily derive the Wiener-Pitt Tauberian theorem [8, p. 163].

The paper is organized as follows. In §2, we introduce the notation and sketch the extension of the Fourier-Wiener transform to abstract Wiener space. In §3 we introduce the $\mathcal{L}_1$-closure theorem. The results on the $\mathcal{L}_1$-closure theorem and the Tauberian theorem are given in the last section.

2. Preliminaries and notation. Let $H$ be a real separable Hilbert-space and suppose $\| \cdot \|_1$ is a measurable norm [5, p. 374]. Then it is known [6] that $\| \cdot \|_1$ is weaker than $\| \cdot \|$ on $H$ and the canonical Gaussian distribution on $H$ induces a Gaussian measure $\mu$ on the Borel subsets of $\mathcal{B}$, the completion of $H$ under $\| \cdot \|_1$. The triple $(\mathcal{B}, \mu, \| \cdot \|_1)$ is called an abstract Wiener space [6] with generating Hilbert space $H$. 

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If $y$ is in $B^*$ (the topological dual of $B$) then the restriction of $y$ to $H$ is continuous on $H$ because $\|\cdot\|_1$ is weaker than $\|\cdot\|$ on $H$. Since $H$ is dense in $B$, the restriction is a one-to-one linear map of $B^*$ into $H^*$.

We shall identify $B^*$ with a subset of $H^*$ and $H^*$ with $H$. Now $B^*$ is dense in $H^* = H$ since $B^*$ separates points of $H$, and hence $B^*$ is dense in $B$. Furthermore, since $B$ and $H$ are separable we have a countable set $\{\alpha_n\}$ of $B^*$ such that $\{\alpha_n\}$ is an orthonormal basis of $H$. Here orthogonality is with respect to inner product in $H$. For each $n$, $(x, \alpha_n)$, $x \in B$, will mean, of course, $\langle x, \alpha_n \rangle$ applied to the vector $x$. Since $\alpha_n \in B^* \subset H$ and $\|\alpha_n\| = 1$, it follows that $\langle \cdot, \alpha_n \rangle$ is a bounded linear functional on $B$ and that it has Gaussian distribution with mean zero and variance one with respect to the measure $\mu$ on $B$. If $h \in H$, we define

$$
(x, h)^\sim = \lim_{n \to \infty} (x, h_n) \quad \text{where} \quad (x, h_n) = \sum_{k=1}^{n} c_k(x, \alpha_k)
$$

and $c_k = (h, \alpha_k)$. We note that $\{(x, \alpha_k)\}$ is a sequence of independent Gaussian functions with mean zero and variance one, and $\sum_{k=1}^{\infty} c_k^2 < \infty$, since $h \in H$ and $\{\alpha_k\}$ is complete orthonormal in $H$. This implies $(x, h)^\sim$ exists for almost all $x \in B$ and it has a Gaussian distribution with mean zero and variance $\|h\|^2$. Furthermore, it is easy to show that $(x, h)^\sim$ equals $(x, h)$ almost everywhere on $B$ if $h \in B^*$, $(x, h)^\sim$ is independent of the complete orthonormal set used in its definition, and finally, if $h_1, h_2, \cdots, h_n$ are orthonormal then $(x, h_1)^\sim, \cdots, (x, h_n)^\sim$ are independent Gaussian functions with mean zero and variance one.

If $f$ is a polynomial in the variables $\{(x, \alpha_k)\}$ then we define the Fourier-Wiener transform $\mathcal{F}$ of $f$ following Cameron and Martin [2], [3] and Segal [9] by

$$
(2.1) \quad \mathcal{F} f(y) = \int_{B} f(\sqrt{2}x + iy) \, d\mu(x) \quad (y \in B).
$$

Here, if $f(x) = g((x, \alpha_1), \cdots, (x, \alpha_N))$ where $g$ is a function of $N$ complex variables, then $f(u+iv) = g((u, \alpha_1)+i(v, \alpha_1), \cdots, (u, \alpha_N)+i(v, \alpha_N))$. Then by [2, p. 491–492] or [9, p. 121], $\mathcal{F}$ is unitary on the class of all polynomials and

$$
(2.2) \quad \mathcal{F}^2 f(y) = f(-y).
$$

Hence in view of the Fourier-Hermite expansion [7, p. 436] one can extend $\mathcal{F}$ to be unitary on $L_2(\mu)$ such that $\mathcal{F} f(y) = f(-y)$. We also remark that $\mathcal{F} f$ can be evaluated as in (2.1) for a much larger class of functionals than the polynomials indicated.

3. Wiener's theorem for $L_2(\mu)$, $0 < p \leq 2$. Let $\lambda$ be the Lebesgue measure on the Borel subsets of the space of real numbers. The classical
theorem of N. Wiener says that any translation invariant (closed) subspace of \( L_2(\lambda) \) consists precisely of those functions whose Fourier transforms vanish on a measurable set. Our purpose in this section is to generalize Wiener’s theorem to \( L_2(\mu) \). Let \( f \in L_2(\mu) \), then for each \( h \in H \), 
\( (U_h f)(y) = f(y + h) \exp\left( -\frac{1}{2}(y, h)^2 - \frac{1}{2} ||h||^2 \right) \) is called the \( L_2 \)-translate of \( f \). It is easy to check by the translation theorem as given in [7, p. 435] that, for each \( h \in H \), \( U_h \) is a unitary operator on \( L_2(\mu) \) onto \( L_2(\mu) \). The main tool of the classical proof is the relation of the Fourier transform of the translate of a function to the Fourier transform of the function itself. The following lemma gives the analogous relation for \( f \in L_2(\mu) \).

**Lemma 3.1.** Let \( f \in L_2(\mu) \), then, for each \( h \in H \),
\[ \mathcal{F}(U_h f)(y) = \exp \left( -\frac{i}{2} (y, h)^2 \right) \mathcal{F}(f)(y) \] with \( \mu \)-measure one on \( B \).

**Proof.** We first assume that \( f \) is a polynomial in some of the variables \( \{(x, \alpha_k): k \geq 1\} \) where \( \{\alpha_k\} \) is completely orthonormal in \( B^* \subseteq H^* \). Then for \( h \in H \) we obtain
\[ \mathcal{F}(f)(y) = \int_B f(\sqrt{2}x + iy) \, d\mu(x) \]
\[ = \int_B f\left( \sqrt{2} \left( x + \frac{h}{\sqrt{2}} \right) + iy \right) \exp \left\{ -\left( x, \frac{h}{\sqrt{2}} \right)^2 - \frac{(h, h)}{4} \right\} d\mu(x) \]
\[ = \int_B f(\sqrt{2}x + iy + h) \exp \left\{ -\frac{1}{2}(\sqrt{2}x + iy, h)^2 - \frac{(h, h)}{4} \right\} d\mu(x) \]
\[ \cdot \exp \left( \frac{i}{2} (y, h) \right) \]
\[ = \int_B (U_h f)(\sqrt{2}x + iy) \, d\mu(x) \cdot \exp \left( \frac{i}{2} (y, h)^2 \right) \]
\[ = \exp \left( \frac{i}{2} (y, h)^2 \right) \mathcal{F}(U_h f)(y). \]
Here the translation by \( h \in H \) is handled as indicated due to the translation theorem [7, p. 435].

For arbitrary \( f \) in \( L_2(\mu) \) we proceed as follows. The operators \( U_h \) and \( \mathcal{F} \) are unitary operators on \( L_2(\mu) \) such that, for each \( h \in H \) and polynomial \( f \) in \( \{(x, \alpha_k)\} \), we have the equation
\[ \mathcal{F}(U_h f) = e_h(\cdot) \mathcal{F} f \]
where \( e_h(\cdot) = \exp\{ -i(\cdot, h)^2/2 \} \) and the equation is understood as \( L_2 \)-equivalence. Since such polynomials are dense in \( L_2(\mu) \) [7, p. 436], the proof follows.
The following is our version of Wiener’s theorem.

**Theorem 3.2.** Let \( \mathcal{W} \) be a translation invariant closed subspace of \( \mathcal{L}_2(\mu) \); i.e., \( U_h \mathcal{W} \subseteq \mathcal{W} \), for each \( h \in H \). Then there exists a measurable subset \( E \) of \( B \) such that \( \mathcal{W} = M_E \) where \( M_E = \{ f \mid f \in \mathcal{L}_2(\mu) \text{ such that } \mathcal{F}f(y) = 0 \text{ a.e. } [\mu] \text{ for all } y \in E \} \). Conversely, each \( M_E \) is translation invariant. Further, \( M_A = M_B \) if and only if \( \mu(A \triangle B) = 0 \).

**Proof.** The converse being obvious from Lemma 3.1 we proceed to the direct part. We note that the proof is basically classical. Let \( \mathcal{W} \) be a translation invariant closed subspace of \( \mathcal{L}_2(\mu) \). Let \( \mathcal{N} = \mathcal{P} \mathcal{W} \). Since \( \mathcal{P} \) is unitary, \( \mathcal{N} \) is a closed subspace of \( \mathcal{L}_2(\mu) \) and in view of Lemma 3.1, \( \mathcal{N} \) is invariant under multiplication by \( e_h(\cdot) \). Let \( P \) be the orthogonal projection of \( \mathcal{L}_2(\mu) \) onto \( \mathcal{N} \). Then \( f - Pf \perp Pg \) for all \( f, g \in \mathcal{L}_2(\mu) \) and since \( \mathcal{N} \) is invariant under multiplication by \( e_h(\cdot) \), we have, for all \( h \in H \),

\[
\int_B (f(x) - (Pf)(x))(Pg)(x)e_{-h}(x) \, d\mu(x) = 0.
\]

Since \( B^* \subset H \), \( B \) is separable, and every (complex) measure on \( B \) is uniquely determined by its Fourier transform, the above equation implies that, for all \( f, g \in \mathcal{L}_2(\mu) \),

\[
f(x)(Pg)(x) = (Pf)(x)(Pg)(x) \text{ a.e. } \mu.
\]

Interchanging the roles of \( f \) and \( g \) we obtain

\[
f(x)(Pg)(x) = g(x)(Pf)(x) \text{ a.e. } \mu.
\]

Taking \( g \equiv 1 \) we get

\[
(Pf)(x) = \varphi(x)f(x) \text{ a.e. } \mu \text{ for all } f \in \mathcal{L}_2(\mu),
\]

where \( \varphi(x) \) is the projection of the function identically one onto \( \mathcal{N} \). But \( P^2 = P \) implies \( \varphi^2 = \varphi \) a.e. \( \mu \). Hence \( \varphi(x) = 0 \) or 1 a.e. \( \mu \) and we let \( E = \{ x; \varphi(x) = 0 \} \). Since \( f \in \mathcal{N} \) iff \( f = Pf = \varphi f \) we see that \( \mathcal{N} \) consists of those functions which vanish at least on \( E \), giving \( \mathcal{W} = M_E \). The uniqueness part being simple is omitted.

**Corollary 3.1 (Wiener [10, p. 267]).** Let \( f \in \mathcal{L}_2(\mu) \) such that \( \mu\{ y; (\mathcal{F}f)(y) = 0 \} = 0 \). Then the linear manifold generated by the \( \mathcal{L}_2 \)-translates of \( f \) is dense in \( \mathcal{L}_p(\mu) \), \( 0 < p \leq 2 \).

**Proof.** Since \( \mathcal{L}_2(\mu) \) is dense in \( \mathcal{L}_p(\mu) \) is suffices to show the theorem in the case \( p = 2 \). Let \( \mathcal{W} \) be the closed subspace of \( \mathcal{L}_2(\mu) \) generated by the \( \mathcal{L}_2 \)-translates of \( f \). Then clearly \( \mathcal{W} \) is translation invariant and since \( \mathcal{F}f(y) \neq 0 \) with \( \mu \)-measure one \( \mathcal{W} = M_\emptyset \) where \( \emptyset \) is the empty set. By definition \( M_\emptyset = \mathcal{L}_2(\mu) \) so the proof is complete.
4. Translation in $L_1(\mu)$ and a Tauberian theorem. The translation operation $U_h f (h \in H)$ as used previously is an isometry on $L_2(\mu)$ onto $L_2(\mu)$, but it is not an isometry in $L_1(\mu)$ unless one has the trivial situation $h=0$. The translation for functions in $L_1(\mu)$ which is an isometry is the following:

\[(4.1) \quad W_h f(x) = f(x + h) \exp\{-(x, h) - \frac{1}{2}(h, h)\} \quad (h \in H).\]

That it is an isometry from $L_1(\mu)$ onto $L_1(\mu)$ follows easily from the translation theorem of Cameron-Martin for this setting [7, p. 435]. As we shall see later it also behaves nicely with respect to convolution.

Now the Fourier-Wiener transform of the translate $W_h f$ (assuming $f$ is in $L_1(\mu)$ so that $\mathcal{F} f$ is defined) does not retain the crucial property of Lemma 3.1. However, if we use a slightly modified Fourier-Wiener transform we can obtain a similar result.

We now define the $L_1$-Fourier-Wiener transform of a function $f \in L_1(\mu)$ by

\[(4.2) \quad \mathcal{F}_1 f(h) = \int_{\mathbb{B}} \exp\{i(x, h)\} f(x) \mu(dx) \quad \text{for} \quad h \in H.\]

In view of the translation theorem [7, p. 435] we have, for each $h_0 \in H$,

\[(4.3) \quad \mathcal{F}_1 (W_{h_0} f)(h) = \exp\{-i(h_0, h)\} \mathcal{F}_1 f(h) \quad \text{for} \quad h \in H.\]

We are now ready to prove a theorem for the $L_1$-transform and $L_1$-translates. As one might guess we make use of Wiener's original theorem in some way. We also point out that the $L_1$-transform in (4.2) is closely related to the Fourier-Wiener transform of Cameron and Martin in [2] which was subsequently modified in [3] becoming the $L_2$-transform.

A function $f$ is a tame function on $B$ if there exist vectors $h_1, h_2, \ldots, h_k \in H$ such that

\[f(x) = \Gamma((x, h_1), \ldots, (x, h_k))\]

where $\Gamma(u_1, u_2, \ldots, u_k)$ is a Borel measurable function on $R_k$. By orthogonalization of $h_1, h_2, \ldots, h_k$ we can always write a tame function in the form

\[(4.4) \quad f(x) = \Psi((x, \varphi_1), \ldots, (x, \varphi_N))\]

where $\varphi_1, \ldots, \varphi_N$ are orthonormal in $H$ and $\Psi$ is Borel measurable on $R_N$. Hence we lose no generality in assuming (4.4) is the case.

**Theorem 4.1.** Let $f \in L_1(\mu)$ be of the form (4.4) and $\mathcal{F}_1 f(h) \neq 0$ for all $h \in H$. Then $\mathcal{W}_p$, the linear manifold generated by the $L_1$-translates of $f$, is dense in $L_p(\mu)$ for $0 \leq p \leq 1$. 

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Proof. Let \( \{q_j; j \geq N+1\} \) be an orthonormal set in \( H \) such that \( \{q_j; j \geq 1\} \) is complete. Then, using the Fourier-Hermite expansion of functions in \( L_2(\mu) \) due to Cameron and Martin [4] and appearing in this generality in Lemma 2.2 of [7], we see that tame functions of the form

\[
L(x) = \Phi[(x, q_1)^{-}, \ldots, (x, q_m)^{-}] \quad (m = 1, 2, \ldots)
\]

where \( L \in L_2(\mu) \) are dense in \( L_2(\mu) \). Since each element in \( L_1(\mu) \) can be approximated in \( L_1 \)-norm by a function in \( L_2(\mu) \) and the \( L_2(\mu) \)-norm is greater than the \( L_1 \)-norm on \( L_2(\mu) \) it follows that functions of the form (4.5) are dense in \( L_1(\mu) \) with respect to the \( L_1 \)-norm. Hence the theorem is proved if the \( L_1 \)-translates approximate any function \( L \in L_2(\mu) \) which is of the form (4.5).

Now any tame function of the form (4.5) with \( m < N \) can be written as a tame function with \( m = N \) by simply multiplying \( \Phi[(u_1, \ldots, u_m)] \) by \( \Phi_1(u_{n+1}, \ldots, u_N) \equiv 1 \) and hence we can assume \( m = N \). On the other hand, if \( m > N \) we then write

\[
f(x) = \Phi[(x, q_1)^{-}, \ldots, (x, q_N)^{-}] \cdot \Phi_2[(x, q_{N+1})^{-}, \ldots, (x, q_m)^{-}]
\]

where \( \Phi_2(u_{N+1}, \ldots, u_m) \equiv 1 \). Hence we can assume without loss of generality that \( m = N \).

By assumption we have for \( h \) of the form \( \sum_{i=1}^{N} a_i q_i \), \( (a_1, a_2, \ldots, a_N) \in R_N \),

\[
\mathcal{F}_1 f(h) = \int_H \exp\{i(x, h)^{-}\} f(x) \mu(dx) \neq 0.
\]

Hence we have

\[
\mathcal{F}_1 f(h) = (2\pi)^{-N/2} \int_{R^N} \exp\left\{ \sum_{i=1}^{N} a_i u_i \right\} \Psi[(u_1, \ldots, u_N)]
\]

\[
\times \exp\left\{ -\frac{1}{2} \sum_{i=1}^{N} u_i^2 \right\} du_1, \ldots, du_N \neq 0.
\]

Thus the ordinary Fourier transform of

\[
\Lambda(v) = (2\pi)^{-N/2} \Psi(v) \exp\{-\frac{1}{2} v \cdot v\} \quad (v \in R_N)
\]

never vanishes on \( R_N \). Here we use \( v \cdot u \) to denote \( \sum_{i=1}^{N} u_i v_i \) if \( u, v \in R_N \). Thus the ordinary translates of (4.7) generate a dense subset of \( L_{1,N} \) where \( L_{1,N} \) denotes the integrable Borel functions with respect to Lebesgue measure on \( R_N \). Let \( L_{1,N}^G \) denote the Borel functions on \( R_N \) which are integrable with respect to the Gaussian density

\[
g(v) = (2\pi)^{-N/2} \exp\{-\frac{1}{2} v \cdot v\} \quad (v \in R_N).
\]

Take an arbitrary tame function \( L \in L_2(\mu) \) of the form (4.5) with \( m = N \).
Then $\Phi(v) \in L^2_{1,N}$ and $\Phi(v)g(v) \in L^1_{1,N}$. Take $\varepsilon > 0$. Then by Wiener's theorem for $L^1_{1,N}$ [8, p. 162] there exist translates $t_1, \ldots, t_k \in \mathbb{R}^N$ and constants $c_1, \ldots, c_k$ such that

$$
\left(4.9\right) \quad \int_{R^N} \left| \sum_{j=1}^{K} c_j \Lambda(v + t_j) - \Phi(v)g(v) \right| \, dv < \varepsilon.
$$

Using (4.7), (4.8), and (4.9) we see that

$$
\left(4.10\right) \quad \int_{R^N} \left| \sum_{j=1}^{K} c_j \Psi(v + t_j) \exp\{ -v \cdot t_j - \frac{1}{2}t_j \cdot t_j \} - \Phi(v)g(v) \right| \, dv < \varepsilon.
$$

Choosing $h_1, \ldots, h_k$ in the subspace of $H$ generated by $\{\varphi_1, \ldots, \varphi_N\}$ and such that $t_j = [(h_j, \varphi_1), \ldots, (h_j, \varphi_N)]$ ($j=1, \ldots, K$), (4.10) then implies

$$
\left(4.11\right) \quad \int_{B} \left| \sum_{j=1}^{K} c_j W_h f(x) - L(x) \right| \, d\mu(x) < \varepsilon.
$$

Hence the tame functions in $L^2_{1}(\mu)$ of the form (4.5) with $m=N$ can be approximated in $L^1$-norm by our $L^1$-translates of $f$. Since $m=N$ represents the general case $\mathbb{R}$ is dense in $L^1_{1}(\mu)$. The proof is now complete since $L^1_{1}(\mu)$ is dense in $L^2_{1}(\mu)$ and the $L^1$-norm dominates the $L^p$-distance, $0 < p \leq 1$.

The function $f \in L^1_{1}(\mu)$ is said to be splittable with respect to the complete orthonormal set $\{\varphi_k\}$ if there exists a sequence of integers $N_1 < N_2 < \cdots$ such that, for each integer $k$,

$$
\left(4.12\right) \quad f(x) = L_k(x) \cdot \Gamma_k(x) \quad \text{a.e. } [\mu]
$$

where $L_k(x) = \Phi_k[\varphi_1, \varphi_2, \cdots, (x, \varphi_N) -]$ and $\Gamma_k$ is $\mathbb{B}_k$ measurable on $B$ where $\mathbb{B}_k$ is the minimal $\sigma$-algebra generated by the functionals $\langle \cdot, \varphi_j \rangle$, $j \geq N_k + 1$.

**Remark 4.1.** Since $\Gamma_k$ is $\mathbb{B}_k$-measurable on $B$ it follows [I, p. 395] that there exists a Borel measurable function defined on the space of all real sequences such that

$$
\Gamma_k(x) = F((x, \varphi_{N_k+1}) -, \cdots) \quad \text{a.e. } [\mu].
$$

Also from (4.12) and the fact that $f \in L^1_{1}(\mu)$ we get that $L_k, \Gamma_k$ are in $L^1_{1}(\mu)$ provided $f \neq 0$.

We say that $f \in L^1_{1}(\mu)$ is negligibly split if $f$ is splittable and for every $\varepsilon > 0$, there exists a $k$ such that

$$
\int_{B} \left| \Gamma_k(x) - 1 \right| \, d\mu(x) < \varepsilon.
$$

It is easy to see that $f$ is then the product of tame functions.
Theorem 4.2. Let \( f \in L_1(\mu) \) be negligibly split with respect to the complete orthonormal bases \( \{ \varphi_k \} \) in \( H \) and assume that \((\mathcal{F}_1 f)(h) \neq 0 \) for all \( h \in H \). Then the linear manifold \( \mathcal{W} \) generated by \( L_1 \)-translates of \( f \) is dense in \( L_2(\mu), 0 < p \leq 1 \).

Proof. In view of the argument given in the proof of Theorem 4.1 it suffices to prove that one can approximate in \( L_1 \)-norms functions \( L \in L_2(\mu) \) of the form (4.5). Take \( \varepsilon > 0 \), and suppose \( L(x) \) is given. Then there exists \( N_k \) such that \( N_k \geq m \) and

\[
\int_B |\Gamma_{N_k}(x) - 1| \, d\mu(x) < \varepsilon.
\]

Again arguing as in Theorem 4.1 we can now assume that \( m = N_k \). Also we have

\[
\mathcal{F}_1 f(y) = \mathcal{F}_1 L_k(y) \mathcal{F}_1 \Gamma_k(y) e^{(y, y)/2}, \quad y \in H.
\]

The above equation holds as indicated since the functionals \( L_k \) and \( \Gamma_k \) are independent (probabilistic sense). Since \( \mathcal{F}_1 f(y) \neq 0 \) we get \( \mathcal{F}_1 L_k(y) \neq 0 \) by (4.14). Now by the proof of Theorem 4.1 there exist constants \( c_1, c_2, \ldots, c_r \) and vectors \( h_1, h_2, \ldots, h_r \) in the subspace of \( H \) generated by \( \{ \varphi_1, \ldots, \varphi_{N_k} \} \) such that

\[
\int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) - L(x) \right| d\mu(x) < \varepsilon.
\]

In view of Remark 4.1 it follows that

\[ \Gamma_k(x + h_j) = \Gamma_k(x) \quad (j = 1, \ldots, r) \]

and hence

\[ W_{h_j} f(x) = [W_{h_j}(L(x))] \cdot \Gamma_k(x) \quad (j = 1, \ldots, r). \]

Now \( \Gamma_k \) and \( L_k \) are independent (probabilistic sense) thus (4.13) and (4.15) imply

\[
\int_B \left| \sum_{j=1}^r c_j W_{h_j} f(x) - \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x)
= \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) [\Gamma_k(x) - 1] \right| d\mu(x)
= \int_B \left| \sum_{j=1}^r c_j W_{h_j} L_k(x) \right| d\mu(x) \cdot \int_B |\Gamma_k(x) - 1| d\mu(x)
\leq \left[ \int_B |L(x)| d\mu(x) + \varepsilon \right] \times \varepsilon.
\]
Combining (4.15) and (4.16) along with $\varepsilon > 0$ being arbitrary completes the proof.

REMARK 4.2. If $f$ is a tame function of the form (4.4) then $f$ is easily seen to be negligibly split with respect to the complete orthonormal basis $\{\varphi_k\}$ where $\{\varphi_1, \ldots, \varphi_N\}$ are as in (4.4). Thus Theorem 4.2 actually implies Theorem 4.1, but we proved both theorems since a direct proof of Theorem 4.2 would involve about the same amount of effort.

For an example of a function $f \in L_1(\mu)$ which satisfies Theorem 4.2 but not Theorem 4.1 consider

$$f(x) = \exp\left\{\sum_{k=1}^{\infty} \lambda_k(x, \varphi_k)^{-2}\right\}$$

where $\{\lambda_k\}$ is a sequence of positive numbers such that $\sum_{k=1}^{\infty} \lambda_k < \frac{1}{2}$ and $\{\varphi_k\}$ is an orthonormal set in $H$. Then $f \in L_1(\mu)$,

$$\mathcal{F}_1 f(h) = \exp\left\{- \sum_{k=1}^{\infty} (h, \varphi_k)^2/2(1 - 2\lambda_k)\right\} \neq 0$$

and $f$ is negligibly split with respect to $\{\varphi_k\}$.

If $f$ and $\varphi$ are measurable functions on $B$ we define the convolution of $f$ and $\varphi$ by the usual formula

$$(4.17) \quad f * \varphi(x) = \int_B f(y) \varphi(y - x) \, d\mu(y).$$

In this setup convolution is not always commutative since the measure $\mu$ is not translation invariant. It does, however, act in a normal way with respect to translation if we use $L_1$-translates and we show this in the next lemma.

**Lemma 4.3.** Suppose $f \in L_1(\mu)$ and $\varphi \in L_\infty(\mu)$. Then

1. $f * \varphi(x)$ exists for each $x$ in $B$,
2. $(W_h f) * \varphi(x) = f * \varphi(x + h)$ for each $x \in B$ and $h \in H$.

**Proof.** Since $\varphi$ is in $L_\infty(\mu)$ and $\varphi(y - x)$ is measurable as a function of $y$ for each $x \in B$ the conclusion of (1) is immediate. Now $f \in L_1(\mu)$ iff $W_h f \in L_1(\mu)$ for each $h \in H$ [7, p. 435] so it follows that $W_h f * \varphi(x)$ exists on $B$ for each $x \in H$. Further,

$$W_h f * \varphi(x) = \int_B W_h f(y) \varphi(y - x) \, d\mu(y)$$

$$= \int_B f(y + h) \exp\{- (x, h)^- - \frac{1}{2}(h, h)^-\} \varphi(y - x) \, d\mu(y)$$

$$= \int_B f(y) \varphi(y - x - h) \, d\mu(y) = f * \varphi(x + h)$$

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where the third equality follows from the translation theorem [7, p. 435]. Hence (2) holds.

We now state a Tauberian theorem for abstract Wiener spaces. Its proof is exactly as in [10, p. 285] if one uses the definition of convolution in (4.17) and the \( L_1 \)-translates. If \( \varphi \) is defined on \( B \) we say \( \lim_{z \to \infty} \varphi(x) = c \) if for each \( \varepsilon > 0 \) there exists a bounded set \( E \) such that \( |\varphi(x) - c| < \varepsilon \) on \( B - E \).

**Theorem 4.3.** Suppose \( f \in L_1(\mu) \) and \( \mathcal{F}_1f(y) \neq 0 \), \( y \in H \). Further, assume \( \varphi \in L_\infty(\mu) \) and \( c \) is a constant such that

\[
\lim_{z \to \infty} f \ast \varphi(x) = c \mathcal{F}_1f(0).
\]

Then, if \( f \) is negligibly split, we have \( \lim g \ast \varphi(x) = c \int_B g(x) \, d\mu(x) \) for all \( g \in L_1(\mu) \).

As a final remark we mention that Pitt's Tauberian theorem, as it appears in [8, p. 163], can also be proved in this setting. Here, however, we would define slowly oscillating in terms of a norm bounded set and a norm bounded neighborhood of zero. Since open subsets of \( B \) have positive \( \mu \)-measure the proof is as in [8].

**References**


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