ON THE RADIUS OF CONVEXITY AND STARLIKENESS OF UNIVALENT FUNCTIONS

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Abstract. In this paper, the converses of the theorems of Bernardi (Trans. Amer. Math. Soc. 135 (1969), 429-446) for the subclasses of univalent functions, namely, starlike functions of order $\beta$, convex functions of order $\beta$ and close-to-convex functions of type $\beta$ and order $\lambda$ have been derived. In particular, these results are sharp and contain the theorems of Padmanabhan (J. London Math. Soc. (2) 1 (1969), 226-231) and Bernardi (Proc. Amer. Math. Soc. 24 (1970), 312-318) as special cases.

1. Introduction. In this paper we study some classes of univalent functions. By $S$ we denote the class of functions $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, which are regular and univalent in the unit disc $D(\{z|<1\}$, while $S^*$ denotes the class of functions in $S$ which map $D$ onto a starlike region with respect to the origin. An equivalent analytic characterization for functions of $S^*$ is well known [8]. By $S^*_\beta$ we denote the class of functions $f(z)$ in $S^*$ having the additional property

$$\text{Re}\left(\frac{zf''(z)}{f'(z)}\right) \geq \beta; \quad z \in D; \quad 0 \leq \beta \leq 1. \quad (1.1)$$

Here $\beta$ is referred as the order of starlike functions $f(z)$ and we identify $S^*_0 = S^*$. The class of functions $f(z)$, which are in $S$ and map $D$ onto a convex domain, is denoted by $C$, while $C^*_\beta$ denotes the class of univalent functions of order $\beta$, if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \beta; \quad z \in D, \quad 0 \leq \beta \leq 1. \quad (1.2)$$

If $f(z) \in S$ and $g(z) \in S^*_\beta$ satisfy the condition

$$\text{Re}\left(\frac{zf'(z)}{g(z)}\right) \geq \lambda; \quad z \in D, \quad 0 \leq \lambda \leq 1, \quad (1.3)$$
then $f(z)$ is said to be a close-to-convex function of order $\lambda$ and type $\beta$. We denote this class by $\Gamma(\lambda, \beta)$. If $\lambda = 0 = \beta$, then $f(z)$ is simply said to be a close-to-convex function with respect to the function $g(z)$ and the class $\Gamma(0, 0)$ is identified by $\Gamma$. This concept of close-to-convex functions is due to Kaplan [3] and its extension appears in the works of Robertson, Libera and others.

Libera [4] in 1965 established the following theorems:

**Theorem A [Libera].** If $f \in S^*$ (or $f \in C$) then the function $F(z) = (2/z) \int_0^z f(t) \, dt \in S^*$ (or $F \in C$).

**Theorem B [Libera].** If $f \in \Gamma$ with respect to $g(z)$ and $F(z) = (2/z) \int_0^z f(t) \, dt$ and $G(z) = (2/z) \int_0^z g(t) \, dt$ then $F \in \Gamma$ with respect to $G$.

Bernardi [1] extended Theorems A and B, and proved the following:

**Theorem C [Bernardi].** If $f(z) \in S^*$ (or $C$), $c = 1, 2, 3, \ldots$,

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n = (c + 1)z^{-c-1} \int_0^z t^{-c-1} f(t) \, dt \quad (a_1 = 1)$$

then $g(z) \in S^*$ (or $C$).

**Theorem D [Bernardi].** If $f \in \Gamma$ with respect to $g$ and

$$F(z) = (c + 1)z^{-c-1} \int_0^z t^{-c-1} f(t) \, dt; \quad G(z) = (c + 1)z^{-c-1} \int_0^z t^{-c-1} g(t) \, dt,$$

c = 1, 2, 3, \ldots, then $F \in \Gamma$ with respect to $G$.

The converse problem of Libera [4] was treated by Livingston [5] who proved the following:

**Theorem E [Livingston].** If $F \in S^*$ then $f(z) = \frac{1}{2}[zF(z)]'$ is starlike for $|z| < \frac{1}{2}$. This result is sharp.

**Theorem F [Livingston].** If $F$ is in $C$, then $f(z) = \frac{1}{2}[zF(z)]'$ is univalent in $D$ and is convex for $|z| < \frac{1}{2}$. This result is sharp.

Bernardi [2] again considered the converse problem of Theorems C and D and obtained the following:

**Theorem G [Bernardi].** If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$,

$$g(z) = \sum_{n=1}^{\infty} \left(\frac{c+1}{c+n}\right) a_n z^n = (c + 1)z^{-c-1} \int_0^z t^{-c-1} f(t) \, dt$$

with $a_1 = 1$ and $c = 1, 2, 3, \ldots$ and $g(z) \in S^*$ (or $C$) then $f(z)$ is starlike (or convex) in the region $|z| < (-2 + (3 + c^2)^{1/2})/(c - 1)$ for $c = 2, 3, 4, \ldots$ and $|z| < \frac{1}{2}$ for $c = 1$. This result is sharp.
THEOREM H [BERNARDI]. Let \( F(z) \) be close-to-convex with respect to \( G(z) \in S^* \) and
\[
f(z) = \frac{1}{1 + c} z^{1-c}[z^c F(z)]', \quad g(z) = \frac{1}{1 + c} z^{1-c}[z^c G(z)]'
\]
then \( f(z) \) is close-to-convex with respect to \( g(z) \) in the region
\[
|z| < \left( -2 + (3 + c^2)^{1/2} \right) / (c - 1)
\]
for \( c=2, 3, 4, \ldots \) and \( |z| < \frac{1}{2} \) if \( c=1 \).

Padmanabhan [7] considered the converse problem of Libera [4] for the class \( S^* \), \( C_\beta \) and \( \Gamma(\lambda, \beta) \), where \( 0 \leq \beta \leq \frac{1}{4} \). In this paper we are mainly concerned with radius of starlikeness and radius of convexity for functions in \( S^*, C_\beta \) and \( \Gamma(\lambda, \beta) \), respectively. In particular, we derive the converses of Theorems G and H of Bernardi for the classes \( S^*, \Gamma(\lambda, \beta) \) and \( C_\beta \). We notice that these results are sharp. With these extensions we deduce the theorems of Padmanabhan [7] also. Incidentally, the proof of our Theorem 1, which includes a theorem of Padmanabhan, is much simpler and can also be adopted for the restricted case considered by him. We state here a lemma due to Bernardi [1] which we shall need,

**Lemma [1, p. 430]**. Let \( f(z) \) and \( g(z) \) be regular in \( |z| < 1 \), \( g(z) \) map \( |z| < 1 \) onto a many sheeted starlike region, \( \alpha, \beta \) real,
\[
\text{Re}\left\{ e^{\beta} f(z) / g(z) \right\} > \alpha
\]
for \( |z| < 1 \), \( g(0) = f(0) = 0 \). Then \( \text{Re}\left\{ e^{\beta} f(z) / g(z) \right\} > \alpha \) for \( |z| < 1 \).

Further, if \( f(z) \in S^* \), \( g(z) = \int_0^1 H(t) \, dt = \int_0^1 t^{\beta-1} f(t) \, dt \), then \( g(z) \) is \((p+1)-\)valent starlike for \( p=1, 2, 3, \ldots \).

2. We shall prove the following:

**Theorem 1.** If \( f(z) = z + \sum_{n=2}^\infty a_n z^n \in S^* \) and
\[
g(z) = \sum_{n=1}^\infty \left( \frac{c + 1}{c + n} \right) a_n z^n = (c + 1)z^{-c} \int_0^z t^{-1} f(t) \, dt,
\]
with \( a_1 = 1 \) and \( c=1, 2, 3, \ldots \), then \( g(z) \in S^* \) and conversely, if \( g(z) \in S^* \) then \( f(z) \) is starlike of order \( \beta \) in the region
\[
|z| < r_0 = \frac{-(2 - \beta) + (3 + \beta^2 + c^2 + 2c\beta - 2\beta)^{1/2}}{c + 2\beta - 1},
\]
\[
eq \frac{1}{2}, \quad \text{if } c = 2, 3, \ldots ,
\]
\[
eq \frac{-(2 - \beta) + (4 + \beta^2)^{1/2}}{2\beta}, \quad \text{if } c = 1 \text{ and } 0 < \beta < 1.
\]
Proof. If \( J(z) = \int_0^z e^{t^2} f(t) \, dt \) then it easily follows on lines similar to those given by Bernardi [1, p. 431] that

\[
\text{Re} \left( \frac{z^{n+1} g'(z)}{J'(z)} \right) = (c + 1) \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq (c + 1) \beta.
\]

From (2.1) and the lemma, the first part of the theorem follows. Conversely, by the hypothesis of the theorem we have

\[
\frac{zg'(z)}{g(z)} = \frac{zJ' - cJ}{J}.
\]

Further, since \( g(z) \) is starlike function of order \( \beta \), so there exists a function \( \omega(z) \) which is regular in the unit disc \( D \) and satisfies the conditions of Schwarz's lemma, such that

\[
\frac{zg'(z)}{g(z)} = \frac{1 - (1 - 2\beta)\omega(z)}{1 + \omega(z)}.
\]

From (2.2) and (2.3) it follows that

\[
f(z) = \frac{[(1 + c) + \{c + (2\beta - 1)\} \omega(z)]J}{[1 + \omega(z)]z^c}.
\]

Differentiating \( f(z) \) logarithmically and simplifying finally we get

\[
\frac{zf'(z)}{f(z)} - \beta = (1 - \beta)
\]

\[
\times \left[ \frac{1 - \omega(z)}{1 + \omega(z)} - \frac{2\omega'(z)}{[1 + \omega(z)][(1 + c) + (c + 2\beta - 1)\omega(z)]} \right].
\]

But

\[
\text{Re} \left( \frac{1 - \omega(z)}{1 + \omega(z)} \right) = \frac{1 - |\omega(z)|^2}{|1 + \omega(z)|^2}
\]

and

\[
\text{Re} \left( \frac{2\omega'(z)}{[1 + \omega(z)][(1 + c) + (c + 2\beta - 1)\omega(z)]} \right) \leq \frac{2 |z| (1 - |\omega(z)|^2)}{(1 - |z|^2) |1 + \omega(z)| |(1 + c) + (c + 2\beta - 1)\omega(z)|}.
\]

The last inequality is obtained by using the following well-known inequality [6, p. 168]:

\[
|\omega'(z)| \leq (1 - |\omega(z)|^2)/(1 - |z|^2).
\]
From (2.4) thru (2.7), we note that \( f(z) \) is starlike of order \( \beta \), if
\[
\frac{2 |z| (1 - |\omega(z)|^2)}{|1 + \omega(z)| (1 - |z|^2) [(1 + c) + (c + 2\beta - 1)\omega(z)]} \leq \frac{1 - |\omega(z)|^2}{|1 + \omega(z)|^2}
\]
or
\[
(2.8) \quad \frac{2 |z|}{1 - |z|^2} \leq (1 + c) \left| 1 + \frac{c + 2\beta - 1}{1 + c} \omega(z) \right| / |1 + \omega(z)|.
\]
Since \( |\omega(z)| \leq |z| \) and \( (c + 2\beta - 1)/(1 + c) \leq 1 \), we have
\[
(2.9) \quad 1 + \frac{c + 2\beta - 1}{1 + c} |z| / (1 + |z|) \leq \left| 1 + \frac{c + 2\beta - 1}{1 + c} \omega(z) \right| / |1 + \omega(z)|.
\]
Hence, by (2.8) and (2.9), we obtain that \( f(z) \in S^* \) if
\[
2 |z| \leq [(1 + c) + (c + 2\beta - 1) |z|] / (1 - |z|)
\]
i.e., if
\[
(1 + c) - 2(2 - \beta) |z| - (c + 2\beta - 1) |z|^2 > 0.
\]
Let
\[
P(|z|) = P(r) = (1 + c) - 2(2 - \beta)r - (c + 2\beta - 1)r^2.
\]
Since \( P(0) = 1 + c \) and \( P'(r) < 0 \), the positive root \( r_0 \) for which \( P(r) > 0 \) must be less than the root of the polynomial \( P(r) = 0 \). This gives the required value of \( r_0 \) and the proof of Theorem 1 is complete.

The following example shows that the result of Theorem 1 is sharp for each \( c \).

**Example 1.** Consider the function
\[
g(z) = z(1 - z)^{-2(1 - \beta)}; \quad 0 \leq \beta \leq 1.
\]
Clearly \( g(z) \in S^* \) and \( f(z) = (z^{1-c}/(1 + c))[zg(z)]' \) imply that
\[
\frac{zf'(z)}{f(z)} = \frac{1 - \beta)(1 + c) + 2(2 - \beta)z - (c + 2\beta - 1)z^2}{1 - z} / [(1 + c) - (c + 2\beta - 1)z]
\]
Thus \( zf'(z)f(z) - \beta = 0 \) for \( z = -r_0 \). Hence \( f(z) \) is not starlike in any disc \(|z| < r_0 \). If \( r > r_0 \), Theorem G of Bernardi now follows by taking \( c = 1, 2, 3, \cdots \) and \( \beta = 0 \). If we take \( c = 1 \) and \( 0 \leq \beta \leq \frac{1}{2} \), then the following theorem of Padmanabhan [7] follows as a corollary to Theorem 1.

**Theorem [Padmanabhan].** Let \( g(z) \in S^* \). Then \( f(z) = \frac{1}{2}[zg(z)]' \) is starlike of order \( \beta \) for
\[
|z| < \left( \frac{(\beta - 2) + (\beta^2 + 4)^{1/2}}{2\beta} \right); \quad 0 \leq \beta \leq \frac{1}{2}.
\]
Theorem 2. If \( g(z) \in C_\beta \), \( f(z) = (1/(1+c))z^{1-c}[z^c g(z)]' \), \( c = 1, 2, 3, \ldots \), then \( f(z) \) is convex of order \( \beta \) in \( |z| < r_0 \) where \( r_0 \) is defined as in Theorem 1. The result is sharp.

Proof. Proof of Theorem 2 follows immediately by using the fact that if \( g(z) \in C_\beta \) then \( zg'(z) \in S_\beta^* \) and conversely (see [8]).

The following example shows that the result of Theorem 2 is sharp.

Example 2. Let \( g(z) = 1 - (1-z)^{2\beta-1}/(2\beta-1) \); \( \beta \neq \frac{1}{2} \) and if \( \beta = \frac{1}{2} \) then \( g(z) = -\log(1-z) \).

If \( \beta \neq \frac{1}{2} \) then by direct computation we find that

\[
f(z) = \frac{(2\beta - 1)c - c(1 - z)^{2\beta-1} + (2\beta - 1)z(1 - z)^{2\beta-2}}{(1 + c)(2\beta - 1)},
\]

\[
f'(z) = \frac{1}{1 + c} \left[ (1 + c) + (1 - c - 2\beta)z \right],
\]

and, therefore

\[
(1 - \beta) + z\frac{zf''(z)}{f'(z)} = \frac{(1 - \beta)[(1 + c) + 2(2 - \beta)z - (c + 2\beta - 1)z^2]}{(1 - z)[(1 + c) - (c + 2\beta - 1)z]}.
\]

Thus the expression \( (1 - \beta) + z\frac{zf''(z)}{f'(z)} \) vanishes for \( z = -r_0 \), hence \( f(z) \) is not convex of order \( \beta \) in any disc \( |z| < r, r > r_0 \). Similarly for \( \beta = \frac{1}{2} \) and \( g(z) = -\log(1-z) \), sharpness of the theorem can be established.

If we take \( c = 1 \) and \( 0 \leq \beta \leq \frac{1}{2} \) in Theorem 2, then the following theorem of Padmanabhan [7] is obtained as a corollary to Theorem 2.

Theorem [Padmanabhan]. Let \( g(z) \in C_\beta \). Then \( f(z) = \frac{1}{2}[zg(z)]' \in C_\beta \) for

\[
|z| < \left[ \frac{(\beta - 2) + (\beta^2 + 4)^{1/2}}{2\beta} \right].
\]

Theorem 3. Let \( f(z) = (z^c/(1+c))[z^c F(z)]' \) and \( g(z) = (z^{1-c}/(1+c)) \times \left[ z^c G(z) \right]' \), \( c = 1, 2, 3, \ldots \), \( G(z) \in S_\beta^* \) and \( F(z) \in \Gamma(\lambda, \beta) \) with respect to \( G(z) \). Then \( f(z) \in \Gamma(\lambda, \beta) \) with respect to the function \( g(z) \in S_\beta^* \) for \( |z| < r_0 \). The result is sharp.

Proof. Since \( G(z) \in S_\beta^* \), from Theorem 1, we have

\[
\text{Re} \left( \frac{zg(z)}{g(z)} \right) \geq \beta \text{ for } |z| < r_0.
\]

Also, since \( F(z) \in \Gamma(\lambda, \beta) \) with respect to \( G(z) \), there exists an analytic function \( \omega(z) \) satisfying the conditions of Schwarz's lemma such that

\[
P(z) \equiv \frac{zF'(z)}{G(z)} = \frac{1 - (1 - 2\lambda)\omega(z)}{1 + \omega(z)}.
\]
But $P(z)$ can also be written as

$$P(z) \int_0^z t^{\varepsilon-1} g(t) \, dt = z^\varepsilon f(z) - c \int_0^z t^{\varepsilon-1} f(t) \, dt. \tag{2.10}$$

By differentiating (2.10) with respect to $z$, we obtain

$$\frac{zf'(z)}{g(z)} = P(z) + \frac{P'(z)}{g(z)} z^{\varepsilon-\varepsilon} \int_0^z t^{\varepsilon-1} g(t) \, dt. \tag{2.11}$$

But

$$\frac{1}{z^\varepsilon g(z)} \int_0^z t^{\varepsilon-1} g(t) \, dt = \frac{G(z)}{cG(z) + zG'(z)}. \tag{2.12}$$

Further, since $G(z) \in S^*_\varepsilon$, there exists an analytic function $V(z)$ satisfying the conditions of Schwarz's lemma, such that

$$\frac{zG'(z)}{G(z)} = \frac{1 - (1 - 2\beta)V(z)}{1 + V(z)} \tag{2.13}.$$

From (2.13) we have

$$\left[ c + \frac{zG'(z)}{G(z)} \right]^{-1} = \left[ \frac{(c + 1) + (c - 1 + 2\beta)V(z)}{1 + V(z)} \right]^{-1}. \tag{2.14}$$

Hence we have from (2.12) and (2.14) that

$$\left[ \int_0^z t^{\varepsilon-1} g(t) \, dt \right] \frac{z^\varepsilon g(z)}{z^\varepsilon g(z)} = \left[ \frac{(c + 1) + (c - 1 + 2\beta)V(z)}{1 + V(z)} \right]^{-1}. \tag{2.15}$$

Thus by using (2.15) we obtain from (2.11) that

$$\Re \left( \frac{zf'(z)}{g(z)} \right) - \lambda \geq \Re \{ P(z) - \lambda \} \times \frac{1}{1 - |z|} \left[ \frac{(c + 1) - 2(2 - \beta) |z| - (c + 2\beta - 1) |z|^2}{(c + 1) + (c + 2\beta - 1) |z|} \right]. \tag{2.16}$$

The inequality (2.16) implies that $f(z) \in \Gamma(\lambda, \beta)$ with respect to the function $g(z)$ if $|z| < r_0$. This completes the proof of Theorem 3. Sharpness of the theorem follows from Theorem 2.

As a corollary to Theorem 3, Theorem H of Bernardi follows by taking $\beta = 0$.

**THEOREM 4.** Let $F(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be regular and have the property $\Re(F'(z)) > \beta$ for $|z| < 1$, $f'(z) = (1/(1 + c)) z^{1-c}[z^c F(z)]', \ c = 1, 2, 3, \cdots$. Then $\Re(f'(z)) > \beta$ for $|z| < r_1 = \left[ -1 + (2 + 2c + c^2)^{1/2} / (1 + c) \right]$. The result is sharp.
Proof. The proof given by Bernardi for theorem [2, p. 317] remains valid except for the following change.

\[(1 + c) \text{Re}\{f'(z) - \beta\} \geq \text{Re}\{P(z) - \beta\} \left[ (1 + c) - \frac{2|z|}{1 - |z|^2} \right].\]

This result is sharp as is seen by the example,

\[F(z) = (2\beta - l)z - 2(1 - \beta)\log(1 - z).\]

References