REPRESENTATIONS OF STRONGLY AMENABLE C*-ALGEBRAS

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Abstract. B. E. Johnson has introduced the concept of a strongly amenable C*-algebra and has proved that GCR algebras and uniformly hyperfinite algebras are strongly amenable. We generalize the well-known Dixmier-Mackey theorem on amenable groups by proving that every continuous representation of a strongly amenable C*-algebra is similar to a *-representation. As an application, we show that every invariant operator range for a Type I von Neumann algebra comes from an operator in the commutant.

Introduction. Let $A$ be a complex Banach algebra. Then a complex Banach space $X$ is a Banach $A$-module if it is a two-sided $A$-module and there exists a positive real number $k$ such that for all $a \in A$ and $x \in X$ we have
\[ \|ax\| \leq k \|a\| \|x\| \quad \text{and} \quad \|xa\| \leq k \|x\| \|a\|. \]

If $X$ is a Banach $A$-module, then the dual space $X^*$ becomes a Banach $A$-module if we define for $a \in A$, $f \in X^*$ and $x \in X$,
\[ (af)(x) = fx(a), \quad (fa)(x) = f(ax). \]

A derivation from $A$ into $X^*$ is a complex linear map $D$ from $A$ into $X^*$ such that $D(ab) = aD(b) + D(a)b$ for all, $a, b \in A$. If $f \in X^*$, the function $\delta(f)$ from $A$ into $X^*$ given by $\delta(f)(a) = af - fa$ is called the inner derivation induced by $f$. We recall that a topological group $G$ is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on $G$ [5]. B. E. Johnson has proved [7, Theorem 2.5] that if $G$ is a locally compact topological group, then $G$ is an amenable group if and only if for all $L^1(G)$-modules $X$ and derivations $D$ of $L^1(G)$ into $X^*$, we have that $D$ is the inner derivation induced by an element of $X^*$. Johnson then defined a Banach algebra $A$ to be amenable if every derivation of $A$ into $X^*$ is inner for all Banach $A$-modules $X$ [7, §5]. Let $A$ be a C*-algebra and let $A_*$ be the C*-algebra obtained by adjoining the identity $e$. Johnson then makes the following definition [7, §7].
**Definition.** The C*-algebra $A$ is strongly amenable if, whenever $X$ is a Banach $A$-module and $D$ is a derivation of $A$ into $X^*$, there is a $f \in \text{co}\{D(u)u^*: u \in U(A_e)\}$ with $D = -\delta(f)$, where $X$ is made into a unital $A_e$-module by defining $xe = ex = x$ for all $x \in X$, $D$ is extended to $A_e$ by defining $D(e) = 0$, $U(A_e)$ is the unitary group of $A_e$, and $\text{co } S$ denotes the $w^*$-closed convex hull of a set $S$ contained in $X^*$.

If $A$ has an identity, $A$ is strongly amenable if and only if the definition is satisfied for all unital $A$-modules $X$ with $A_e$ replaced throughout by $A$ [7, Proposition 7.2]. Johnson proved that the class of strongly amenable C*-algebras contains all GCR C*-algebras, and all uniformly hyperfinite C*-algebras [7, Theorem 7.9 and Proposition 7.6]. He also proved [7, Proposition 7.8] that if $G$ is an amenable locally compact group, then the $C^*$-group algebra [2, 13.9] is strongly amenable.

It is a well-known theorem, due to Dixmier [1], that every uniformly bounded strongly continuous representation of an amenable group on a Hilbert space is similar to a unitary representation. The main result of this paper is that every continuous representation of a strongly amenable C*-algebra on a Hilbert space is similar to a *-representation.

**The main results.**

**Lemma 1.** Let $A$ be a strongly amenable C*-algebra, $X$ a Banach $A$-module, and let $C = \{f \in X^*: af = fa$ for all $a \in A\}$. Then for all $f \in X^*$, $C \cap \text{co}\{ufu^*: u \in U(A_e)\}$ is nonempty.

**Proof.** The proof is a generalization of a proof of Johnson's [7, just before Proposition 7.14]. Let $f$ be in $X^*$ and let $\delta(f)$ be the inner derivation defined by $f$. Then there is a $g \in \text{co}\{\delta(f)(u)u^*: u \in U(A_e)\}$ such that $\delta(f) = -\delta(g)$. But $\delta(f)(uu^*) = uf - f$, so $f + g \in \text{co}\{ufu^*: u \in U(A_e)\}$. Also, $\delta(f)(a) = -\delta(g)(a)$ for all $a \in A$, so $af - fa = ga - ag$, or $a(f + g) = (f + g)a$. Thus $f + g \in C$.

We recall that a linear functional $f$ on a C*-algebra $A$ is called central if $f(ba) = f(ab)$ for all $a, b \in A$.

**Corollary 1.** If $A$ is strongly amenable C*-algebra with identity, then $A$ has positive central functionals of norm one and hence $A$ has nonzero factor *-representations of finite type.

**Proof.** If $f$ is any state of $A$, then by Lemma 1 (with $X = A$) there is a $g \in \text{co}\{ufu^*: u \in U(A)\}$ such that $g(ab) = g(ba)$ for all $a, b \in A$. Then $g$ is clearly positive and $g(e) = 1$. The rest follows from [2, 6.8].

Since GCR algebras have only Type I *-representations and are strongly amenable, the following corollary is immediate.

**Corollary 2.** If $A$ is a GCR algebra with identity, then $A$ has nonzero finite-dimensional *-representations.
Let $B(H)$ be the bounded operators on Hilbert space $H$ and let $K(H)$ be the compact operators on $H$.

**Corollary 3.** The Calkin algebra $B(H)/K(H)$ (for $H$ separable Hilbert space) is not strongly amenable, so $B(H)$ is not strongly amenable.

**Proof.** Let $p$ be a projection in $B(H)$ with infinite-dimensional range and null-space, so that $p$ is equivalent to the identity in $B(H)$. Then let $\tilde{p}$ be the image of $p$ in $B(H)/K(H)$. Since $B(H)/K(H)$ is simple [10, p. 291] it is clear that in any $\ast$-representation $\tilde{p}$ will be a nontrivial projection equivalent to the identity. Thus every $\ast$-representation is infinite and Corollary 1 implies that $B(H)/K(H)$ is not strongly amenable. Since quotients of strongly amenable $C^\ast$-algebras are strongly amenable [7, 7.3], $B(H)$ is not strongly amenable.

For $A$ a $C^\ast$-algebra, let $A \hat{\otimes} A$ be the completion of $A \otimes A$ in the greatest cross-norm. Then $(A \hat{\otimes} A)^\ast$ is the space of bounded bilinear functionals on $A \times A$ [6, p. 30]. We see that $A \hat{\otimes} A$ becomes a Banach $A$-module if we define for $a, b, c \in A$,

$$a(b \otimes c) = ab \otimes c, \quad (b \otimes c)a = b \otimes ca.$$ 

Hence $(A \hat{\otimes} A)^\ast$ becomes a Banach $A$-module under the dual action: if $f \in (A \hat{\otimes} A)^\ast$ and $a, b, c \in A$,

$$(af)(b \otimes c) = f(b \otimes ca), \quad (fa)(b \otimes c) = f(ab \otimes c).$$

Let $C = \{f \in (A \hat{\otimes} A)^\ast : af = fa \text{ for all } a \in A\}$.

We can also make $A \hat{\otimes} A$ and $(A \hat{\otimes} A)^\ast$ into Banach $A$-modules by defining for $f \in (A \hat{\otimes} A)^\ast$ and $a, b, c \in A$:

$$a \circ (b \otimes c) = b \otimes ac, \quad (b \otimes c) \circ a = ba \otimes c,$$

$$\quad (a \circ f)(b \otimes c) = f(ba \otimes c), \quad (f \circ a)(b \otimes c) = f(b \otimes ac).$$

The map $T$ in the following proposition takes the place of the invariant mean which is present in amenable groups.

**Proposition 1.** Let $A$ be a strongly amenable $C^\ast$-algebra with identity $e$. Then there exists a linear map $T : (A \hat{\otimes} A)^\ast \to C$ such that:

(a) $T(f) \in \co\{ufu^*: u \in U(A)\}$ for all $f \in (A \hat{\otimes} A)^\ast$, and

(b) $T(a^\ast f) = a^\ast T(f)$ and $T(f^\ast a) = T(f)^\ast a$ for all $a \in A$ and $f \in (A \hat{\otimes} A)^\ast$.

**Proof.** Let $Y = (A \hat{\otimes} A)^\ast \hat{\otimes} (A \hat{\otimes} A)$ be made into an $A$-module with operations $(f \otimes t)a = f( \otimes (ta))$, $a(f \otimes t) = f( \otimes (at))$, for $f \in (A \hat{\otimes} A)^\ast$, $t \in (A \hat{\otimes} A)$, and $aeA$. Let $Z$ be the closed submodule of $Y$ spanned by elements of the form $(a^\ast f) \otimes t - f( \otimes (ta))$ and $(f \ast a) \otimes t - f( \otimes (at))$. Define an element $F$ of $Y^\ast$ by $F(f \otimes t) = f(t)$, $f \in (A \hat{\otimes} A)^\ast$, $t \in (A \hat{\otimes} A)$. Then $F$ is zero on $Z$, so if we let $X = Y/Z$, we can regard $F$ as an element of $X^\ast$. Then apply Lemma 1 to get an
element $T_0$ of $X^*$ such that $aT_0 = T_0a$ for all $a$ in $A$, and $T_0 \in \text{co}(uFu^*: u \in U(A))$. Then define a bounded endomorphism $T$ of $(A \otimes A)^*$ by $T(f)(t) = T_0((f \otimes t)^{-1})$, where $(f \otimes t)^{-1}$ means the coset of $f \otimes t$ in $X$. Then clearly $T$ maps into $C$ and satisfies (b). An application of the strong separation theorem shows that (a) is satisfied.

We use the existence of the function $T$ to prove our main result.

**Theorem 1.** Every continuous representation of a strongly amenable C*-algebra $A$ on a Hilbert space is similar to a $*$-representation.

**Proof.** If $V: A \rightarrow B(H)$ is a continuous representation of $A$ on $H$ (i.e., $V$ is a continuous algebra homomorphism of $A$ into the bounded operators on a Hilbert space $H$), then $V$ may be extended to the algebra $A_e$ by defining $V'(a, \lambda) = V(a) + \lambda$. Then $V'$ is a continuous representation of $A_e$, and $A_e$ is strongly amenable if $A$ is strongly amenable [7, 7.3]. So it suffices to assume that $A$ has an identity and $V(e) = e$. Let $x, y \in H$ and define $f_{x,y} \in (A \otimes A)^*$ by

$$f_{x,y}(a \otimes b) = (V(a)x, V(b^*)y).$$

We then define a new inner product on $H$ by $(x, y)_1 = T(f_{x,y})(e \otimes e)$. Then $(x, y)_1$ is a bounded sesquilinear form on $H$, so there is a bounded operator $R \in B(H)$ such that $(Rx, y) = T(f_{x,y})(e \otimes e)$ for all $x, y \in H$. Now $T(f_{x,y}) \in \text{co}(uf_{x,y}u^*: u \in U(A))$ and for $u \in U(A)$ we have

$$uf_{x,y}u^*(e \otimes e) = f_{x,y}(u^* \otimes u) = \|V(u^*)x\|^2.$$

Then $\|x\|^2 = \|V(u)V(u^*)x\|^2 \leq \|V\|^2 \|V(u^*)x\|^2 \leq \|V\|^4 \|x\|^2$, so

$$\|V\|^2 \|x\|^2 \leq \|V(u^*)x\|^2 \leq \|V\|^2 \|x\|^2$$

for all $u \in U(A)$ and all $x \in H$. Thus

$$\|V\|^2 \|x\|^2 \leq T(f_{x,y})(e \otimes e) \leq \|V\|^2 \|x\|^2.$$

Hence $R$ is a positive invertible operator. Let $S$ be the (positive) square root of $R$, then $(Sx, Sy) = T(f_{x,y})(e \otimes e)$. We now show that the representation $V'(a) = SV(a)S^{-1}$ is a $*$-representation. Let $u \in U(A), z \in H, a, b \in A$ and compute:

$$f_{V(u)z, V(u)z}(a \otimes b) = (V(a)V(u)z, V(b^*)V(u)z) = f_{z,z}(au \otimes u^*b),$$

so that $f_{V(u)z, V(u)z} = uf_{z,z} \circ u^*$. Then we have

$$(V(u)z, V(u)z)_1 = T(f_{V(u)z, V(u)z})(e \otimes e) = T(u \circ f_{z,z} \circ u^*)(e \otimes e) = T(f_{z,z})(u \otimes u^*) = (u^*T(f_{z,z})u)(e \otimes e) = T(f_{z,z})(e \otimes e) = (z, z)_1.$$
Finally,

\[(V'(u)z, V'(u)z) = (SV(u)S^{-1}z, SV(u)S^{-1}z) = (V(u)S^{-1}z, V(u)S^{-1}z)_1 = (S^{-1}z, S^{-1}z)_1 = (z, z).\]

In the above computation we used the fact that \( T \) satisfies both parts of property (b) in Proposition 1, and we used the fact that \( T \) maps into \( C \). So \( V' \) is \( * \)-preserving on the unitary elements of \( A \), and since every element of \( A \) is a linear combination of four unitary elements, we have that \( V' \) is a \( * \)-representation.

**Applications and remarks.** J. Dixmier has asked the following question: Let \( A \subset B(H) \) be a von Neumann algebra. Suppose \( b \in B(H) \) is such that \( b(H) \) is invariant for \( A' = \text{the commutant of } A \). Then does there exist an operator \( a \in A \) with \( a(H) = b(H) \)? Using Theorem 1 we can answer this question in the affirmative in a special case.

**Proposition 2.** Let \( A \subset B(H) \) be a strongly amenable \( C^* \)-algebra. Let \( b \in B(H) \) be such that \( b(H) \) is invariant under \( A \). Then there is an operator \( a \in A' \) with \( a(H) = b(H) \).

**Proof.** The proof is almost exactly the same as a proof of C. Foias [4, Lemma 8] which uses Dixmier's theorem for representations of amenable groups. Since \( (bb^*)^{1/2} \) and \( b \) have the same range, we may assume that \( b \geq 0 \). Now define a representation \( V: A \to B(H) \) as follows: For \( d \in A \), \( d(b(H)) \subset b(H) \), so if \( x \in H \), there is a unique vector \( y \in \text{null}(b)^\perp \) such that \( (db)(x) = by \). Let \( V(d)x = y \). Then \( V(d) \) is well defined, \( V(d) \) is clearly linear, and \( V(d) \) has closed graph. So \( V(d) \in B(H) \). Also \( V(d)(H) \subset \text{null}(b)^\perp = (b(H))^\perp \) and \( db = bV(d) \). It is then easily seen that \( V \) is linear with closed graph and \( V(cd) = V(c)V(d) \). Hence \( V: A \to B(H) \) is a continuous representation, so by Theorem 1, there is a positive invertible \( S \in B(H) \) such that \( U(d) = SV(d)S^{-1} \) is a \( * \)-representation. Let \( S^{-1}b = ua \) be the polar decomposition of \( S^{-1}b \). Then \( bS^{-1} = au^* \), so \( b = au^*S \) and \( b(H) \subset a(H) \). Also \( bS^{-1}u = a \), so \( a(H) \subset b(H) \) and \( a(H) = b(H) \). We will thus be finished when we show that \( a \in A' \). Let \( d \in A \). Then \( db = bV(d) = bS^{-1}U(d)S \), so \( dbS^{-1} = bS^{-1}U(d) \) and \( dau^* = au^*U(d) \), or \( da = au^*U(d)u \) for all \( d \in A \). So we also have \( d*a = au^*U(d^*)u \), and since \( U \) is a \( * \)-representation we have \( d*a = au^*V(d^*)u \), or \( ad = u^*U(d)ua \). Then \( a^2 = au^*U(d)ua = da^2 \). So \( a^2 \in A' \) and \( a \in A' \).

The following corollary is then immediate.

**Corollary 4.** Let \( A \subset B(H) \) be a von Neumann algebra such that \( A' \) contains a weakly dense strongly amenable \( C^* \)-algebra \( B \). Suppose \( b \in B(H) \) is such that \( b(H) \) is invariant for \( B \). Then \( b(H) \) is also invariant for \( A' \) and there is an \( a \in A \) with \( a(H) = b(H) \).
By [11] every Type I von Neumann algebra on a separable Hilbert space contains a weakly dense GCR algebra. So since GCR algebras are strongly amenable, the above corollary covers the case when $A$ is a Type I von Neumann algebra on a separable Hilbert space, as well as the case when $A'$ is a hyperfinite von Neumann algebra.

We close by remarking that Ehrenpreis and Mautner [3] give an example of a GCR group $G$ which has a uniformly bounded representation on a Hilbert space that is not similar to a unitary representation. Our Theorem 1 then implies that the induced representation of $L^1(G)$ is not continuous in the $C^*$-algebra norm. The author does not know of an example of a continuous representation of a $C^*$-algebra on a Hilbert space that is not similar to a $^*$-representation.

REFERENCES


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