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NOTE RELATING BOCHNER INTEGRALS AND REPRODUCING KERNELS TO SERIES EXPANSIONS ON A GAUSSIAN BANACH SPACE

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ABSTRACT. Fernique's recent proof of finiteness of positive moments of the norm of a Banach-valued Gaussian random vector $\mathfrak{X}$ is used to prove rth mean convergence of reproducing kernel series representations of $\mathfrak{X}$. Embedding of the reproducing kernel Hilbert space into the Banach range of $\mathfrak{X}$ is explicitly given by Bochner integration. This work extends and clarifies work of Kuelbs, Jain and Kallianpur.

Fernique [2] has recently proved in a most elementary way that $\lim_{\alpha\to 0+} E \exp \alpha\|\mathfrak{X}\|_B^2 < \infty$ for every centered Gaussian random vector $\mathfrak{X}$ taking values in a real and separable Banach space B. As will be shown below, this result can be used to provide a dramatically simple proof of the strong convergence of certain representations of $\mathfrak{X}$ by a series in B, as given by Kuelbs [4] and Jain-Kallianpur [3]. The role of reproducing kernel Hilbert spaces in such representations is sharply revealed by this approach.

In this paper, B is a real and separable Banach space, B^* its topological dual, $\mathscr{B}$ is the σ-algebra generated by the open subsets of B, and P is a probability measure on $\mathscr{B}$ for which the induced distributions of the random variables $x^* \in B^*$ are all Gaussian with zero means.

Suppose that in addition to being a Banach space, B is also a subset of the set of real functions on a set T (distinct points of B also being distinct as real functions on T), and that for each $t \in T$ the evaluation mapping $\mathfrak{X}_t$ defined by $\mathfrak{X}_t(x) = x(t)$, $x \in B$, is continuous on B. For example, if T is taken equal to B^*, each $x \in B$ may be viewed as the continuous linear evaluation function on B^* defined by $x(x^*) = x^*(x)$, $x^* \in B^*$. Let $\mathscr{L}$ denote P quadratic-mean closure of $\{\mathfrak{X}_t, t \in T\}$ viewed as a Hilbert subspace of

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$L_2(B, \mathscr{B}, P)$, and denote by $H(R)$ the reproducing kernel Hilbert space of

$$R(s, t) = \int_B x(s)x(t)P(dx), \qquad (s, t) \in T \times T.$$

It is well known (e.g. see [5]) that $\mathscr{L}$ is isometrically isomorphic to $H(R)$ under the linear extension of the mapping $\mathfrak{X}_t \leftrightarrow R(t, \cdot)$, $t \in T$, and that $H(R)$ is characterized as the unique Hilbert space of real functions on T containing the sections $R(t, \cdot)$, $t \in T$, and satisfying $f(t) = (f, R(t, \cdot))_{H(R)}$ for every $t \in T$, $f \in H(R)$. The latter is termed the *reproducing property* and this isomorphism of $\mathscr{L}$ with $H(R)$ is termed the *natural isomorphism.*

PROPOSITION. *Consequent to the preceding assumptions*:

(a) *If $L \in \mathscr{L}$, the element of $H(R)$ to which L corresponds under the natural isomorphism is given by the convergent Bochner integral*

$$x_L = \int_B L(x)xP(dx) \in H(R) \subset B$$

which may be calculated pointwise

$$x_L(t) = \int_B L(x)x(t)P(dx), \qquad t \in T.$$

(b) *If $L_1, L_2, \cdots$ are a complete orthonormal set for $\mathscr{L}$, then for every $r \geqq 1$, as $n \to \infty$,*

$$\int_B \left\| x - \sum_{k=1}^{k=n} L_k(x)x_{L_k} \right\|_B^r P(dx) \to 0.$$

In particular, this is a series representation by $H(R)$ functions, the series converges almost surely, and closure of $H(R)$ in B gives the support of P.

PROOF. Suppose $L \in \mathscr{L}$. Let $v(x) = L(x)x$, $x \in B$. Then v is a Banach-valued random vector and for every $r \geqq 1$,

$$\int_B \|v(x)\|_B^r P(dx) \leqq \left(\int_B L^{2r}(x)P(dx) \int_B \|x\|_B^{2r} P(dx) \right)^{1/2}.$$

Since L is the P quadratic-mean limit of P-Gaussian random variables, L is itself P-Gaussian and $\int_B L_B^{2r}(x)P(dx) < \infty$. By Fernique's result quoted earlier, $\int_B \|x\|_B^{2r} P(dx) < \infty$. Therefore $\int_B \|v(x)\|_B^r P(dx) < \infty$ for each $r \geqq 1$.

(a) Suppose $L \in \mathscr{L}$. Taking $r = 1$ in the above we conclude [1] that the Bochner integral $x_L = \int_B L(x) \times P(dx) \in B$ exists.[2] For each $t \in T$, continuity

[2] In fact $\|x_L\|_B \leqq \|x_L\|_{H(R)} \|P\|$, where $\|P\|^2 = \int_B \|x\|_B^2 P(dx) < \infty$, follows immediately once it is established that x_L is companion to L under the natural isomorphism.

of the linear $\mathfrak{X}_t$ enables passage of $\mathfrak{X}_t$ inside the Bochner integral. Let $\hat{L}\in H(R)$ correspond to L under the natural isomorphism. Then, for each $t\in T$,

$$\hat{L}(t) = (\hat{L}, R(t, \cdot))_{H(R)} = (L, \mathfrak{X}_t)_{\mathscr{L}} = x_L(t).$$

(b) Suppose $L_1, L_2, \cdots$ are a complete orthonormal set for $\mathscr{L}$. These $L_1, L_2, \cdots$ have the P-law of independent and identically distributed Gaussian random variables with means zero and variances unity. If $n>0$, $L\in\langle L_1, \cdots, L_n\rangle_{\mathscr{L}}$ (the submanifold of $\mathscr{L}$ spanned by $L_1, \cdots, L_n$) then

$$\begin{aligned} L\left(\sum_{k=1}^{k=n} L_k x_{L_k}\right) &= \sum_{k=1}^{k=n} L_k L(x_{L_k}) \\ &= \sum_{k=1}^{k=n} L_k \int_B L_k(x)L(x)P(dx) = L \quad \text{a.e. } P. \end{aligned}$$

Therefore

$$\sum_{k=1}^{k=n} L_k x_{L_k}, \qquad \mathscr{F}_n = \sigma\{L_1, \cdots, L_n\}, \qquad n \geqq 1,$$

is a strong martingale in the sense of [1]. That is, for $n\geqq 1$,

$$(A \in \mathscr{F}_n) \Rightarrow \left(\int_A \sum_{k=1}^{k=n} L_k(x)x_{L_k}P(dx) = \int_A xP(dx)\right).$$

Since $\mathscr{B}\subset$ the P-completion of $\sigma(\bigcup_n \mathscr{F}_n)$, we conclude from [1, Theorem 1] that, as $n\to\infty$,

$$\left\| \mathfrak{X} - \sum_{k=1}^{k=n} L_k(x)\mathfrak{X}_{L_k} \right\|_B \to 0 \quad \text{a.e. } P.$$

This implies that closure of $H(R)$ in B gives the support of P. For if $x\in B$ and every B open neighborhood of x has positive probability then there are sums of the type $\sum_{k=1}^{k=n} L_k(x_1)x_{L_k}$ (for $n\geqq 1$, and $x_1\in B$) of arbitrary B-closeness to x, and (by (a)) belonging to $H(R)$. If, on the other hand, there is an $\varepsilon>0$ and $L\in\mathscr{L}$ for which an ε-radius B-sphere containing x_L has P-probability zero, it follows from mutual absolute continuity of Gaussian measures under translation by $H(R)$ functions (e.g. see [5]) that an ε-radius B-sphere containing the origin of B has zero P-probability. For the purpose of proving this impossible we may as well assume this sphere is centered at the origin. Then choose n sufficiently large so that with positive P-probability $\|\mathfrak{X}\|_B - \|\sum_{k=1}^{k=n} L_k(\mathfrak{X})x_{L_k}\|_B < \varepsilon$. Since the latter event involves only the tail of this series in mutually independent summands, it suffices to prove that $\|\sum_{k=1}^{k=n} L_k(\mathfrak{X})x_{L_k}\|_B$ has positive probability of being in every interval about zero. By footnote 1 however

$$\left\| \sum_{k=1}^{k=n} L_k(\mathfrak{X})x_{L_k} \right\|_B^2 \leqq \|P\|^2 \sum_{k=1}^{k=n} L_k^2(\mathfrak{X}) < \infty$$

and the P-probability that $\sum_{k=1}^{k=n} L_k^2(\mathfrak{X})<\delta$ is positive for every $\delta>0$. Finally, for every $n\geqq 1$, and $r\geqq 1$,

$$\left(\int_B \left\| \sum_{k=1}^{k=n} L_k(x)x_{L_k}P(dx)\right\|_B^r P(dx)\right)^{1/r} \leqq \sum_{k=1}^{k=n}\left(\int_B \|L_k(x)x_{L_k}\|_B^r\, P(dx)\right)^{1/r} < \infty.$$

From [**1**, Theorem 1] we also conclude that, as $n\to\infty$,

$$\int_B \left\|x - \sum_{k=1}^{k=n} L_k(x)x_{L_k}\right\|_B^r P(dx) \to 0. \quad \square$$

For applications of the Proposition see [**3**], [**4**]. In [**6**], Walsh applied the Chatterji Theorem [**1**, Theorem 1] in much the same way as here, to the Wiener measure case. The reproducing kernel representation (a) was not given however. More recently, Kuelbs [**4**] proved the existence of a representation of type (b) bypassing the Chatterji result, and hence avoiding the question of integrability of $\|\mathfrak{X}\|_B$. The role of reproducing kernels was not discussed. Finally, Jain and Kallianpur [**3**] gave still another proof bypassing the Chatterji result, showing the existence of certain embeddings of the reproducing kernel Hilbert space into B. The representation (a) was not given. Neither [**3**] nor [**4**] discuss the convergence in rth mean.

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