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GENERATING FUNCTIONS FOR JACOBI AND RELATED POLYNOMIALS

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ABSTRACT. In the present paper, we have established the following relation involving Kampe de Feriet's double hypergeometric function of superior order

$$F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & b, -b' & -xy, -y \\ q & c_1, \cdots, c_q & \\ 0 & \cdots & \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{(-1)^n(-b')_n}{n!} {}_2F_1\left[\begin{array}{c} -n, b; \\ 1 + b' - n; \end{array} x\right] y^n,$$

which yields a number of interesting generating formulae for Jacobi and related polynomials.

A large number of special cases have been also discussed.

1. **Introduction.** In this paper, we establish some generating relations of general nature for Jacobi, Gegenbauer and Legendre polynomials, employing an arbitrary sequence of parameters. Certain particular cases yield several interesting generating relations for Laguerre polynomials also.

The results proved here lead to certain generalizations of the various well known formulae due to Brafman [**2**], Carlitz [**3**], Feldheim [**6**], Jain [**7**], [**8**], Manocha [**9**], [**10**] and several other results also.

The Kampe de Feriet's double hypergeometric function of superior order [**1**, p. 150] is defined by

(1.1)

$$F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ q & b_1, b_1'; \cdots; b_q, b_q' & x, y \\ r & c_1, \cdots, c_v & \\ s & d_1, d_1'; \cdots; d_s, d_s' & \end{array}\right] = \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{m+n} \prod_{j=1}^{q} \{(b_j)_m (b_j')_n\}}{\prod_{j=1}^{r} (c_j)_{m+n} \prod_{j=1}^{s} \{(d_j)_m (d_j')_n\}} \frac{x^m y^n}{m!\, n!},$$

$$|x| < 1, \quad |y| < 1.$$

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The Jacobi polynomials [**11**, p. 254 (1)] are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n;\\ 1+\alpha;\end{matrix}\ \frac{1-x}{2}\right]. \tag{1.2}$$

The Gegenbauer polynomials [**11**, p. 280 (19)] are defined by

$$C_n^b(x) = \frac{(b)_n(2x)^n}{n!}\,{}_2F_1\left[\begin{matrix}\frac{-n}{2},\ \frac{-n+1}{2};\\ 1-b-n;\end{matrix}\ \frac{1}{x^2}\right], \tag{1.3}$$

which reduces to Legendre polynomials $P_n(x)$ [**11**, p. 166 (4)] on putting $b=\frac{1}{2}$.

The Laguerre polynomials [**11**, p. 200 (1)] are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_1F_1(-n;\,1+\alpha;\,x). \tag{1.4}$$

2. We require the following formulae to prove our results:

$$\begin{aligned}&{}_2F_1[a,\,b;\,1+a-b;\,z]\\ &\quad = (1+z)^{-a}\,{}_2F_1\left[\frac{a}{2},\ \frac{a+1}{2};\ 1+a-b;\ 4z(1+z)^{-2}\right],\end{aligned} \tag{2.1}$$

which is formula (2) in [**11**, p. 66].

(2.2)

$${}_2F_1\left[\begin{matrix}a,\ b;\\ 2b;\end{matrix}\ \frac{4x}{(1+x)^2}\right] = (1+x^2)^{-a}(1+x)^{2a}\,{}_2F_1\left[\begin{matrix}\frac{a}{2},\ \frac{a+1}{2};\\ b+\frac{1}{2};\end{matrix}\ \frac{4x^2}{(1+x^2)^2}\right],$$

which is a formula on p. 66 in [**11**].

$$F_1(\alpha,\beta,\beta',\beta+\beta',x,y) = (1-y)^{-\alpha}\,{}_2F_1\left(\alpha,\beta;\beta+\beta';\frac{x-y}{1-y}\right), \tag{2.3}$$

where F_1 is Appell's first function [**4**, p. 224]. This is formula (1) in [**4**, p. 238].

In (2.3), replacing x by x/α, y by y/α and making $\alpha\to\infty$, we get

$$\Phi_2(\beta,\beta',\beta+\beta',x,y) = e^y\,{}_1F_1(\beta;\beta+\beta';x-y), \tag{2.4}$$

where Φ_2 is Horn's function [**4**, p. 225, (21)].

$$P_n^{(\alpha-n,-\alpha-\gamma-n)}(x) = \left(\frac{1+x}{2}\right)^n P_n^{(\alpha-n,\gamma-1)}\left(\frac{3-x}{1+x}\right), \tag{2.5}$$

which is a relation in [**12**, p. 63].

3. Let us consider

$$\sum_{m,n=0}^{\infty} C_{m+n} \frac{(b)_m(-b')_n}{m!\,n!}(-xy)^m(-y)^n$$

$$(3.1a) \qquad = \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n \frac{(b)_m(-b')_{n-m}}{m!\,(n-m)!}(-y)^n x^m$$

$$= \sum_{n=0}^{\infty} C_n \frac{(-1)^n(-b')_n}{n!}\, {}_2F_1\left[\begin{matrix}-n, b; \\ 1+b'-n;\end{matrix}\; x\right] y^n,$$

where $\{C_n\}$ is an arbitrary sequence (subject to convergence conditions).

In (3.1a), replacing the arbitrary sequence $\{C_{m+n}\}$ by

$$\left[\prod_{j=1}^{p}(a_j)_{m+n}\right] \Big/ \left[\prod_{j=1}^{q}(c_j)_{m+n}\right]$$

(a, c arbitrary parameters), and making use of (1.1), we obtain the following relation:

$$F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & b, -b' & -xy, -y \\ q & c_1, \cdots, c_q & \\ 0 & \cdots & \end{array}\right]$$

$$(3.1) \qquad = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n(-1)^n(-b')_n}{\prod_{j=1}^{q}(c_j)_n n!}\, {}_2F_1\left[\begin{matrix}-n, b; \\ 1+b'-n;\end{matrix}\; x\right] y^n.$$

In (3.1) replacing x by $(1-x)/2$ and applying (1.2), we have a generating relation for Jacobi polynomials

$$(3.2) \quad F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & b, -b' & \dfrac{(x-1)y}{2}, \\ q & c_1, \cdots, c_q & -y \\ 0 & \cdots & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(c_j)_n} P_n^{(b'-n,b-b'-1)}(x) y^n,$$

$$|(x-1)y/2| < 1, \quad |y| < 1.$$

Replacing x by $(3-x)/(1+x)$, y by $(1+x)y/2$ and b by $-b$ and making

use of (2.5), the above formula (3.2) assumes the following form:

$$(3.3)\qquad F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & -b, -b' & \dfrac{(1-x)y}{2}, \\ q & c_1, \cdots, c_q & \\ 0 & \cdots & \dfrac{-(1+x)y}{2} \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (c_j)_n} P_n^{(b'-n,b-n)}(x) y^n,$$

$$\left|\frac{(1-x)y}{2}\right| < 1, \quad \left|\frac{(1+x)y}{2}\right| < 1.$$

Further, supposing $u=t-(t^2-1)^{1/2}$, $v=1/u=t+(t^2-1)^{1/2}$, replacing x by u^2, y by $-y/u$ and b' by $-b$, and making use of (2.1) and (1.3), the relation (3.1) yields the following generating relation for Gegenbauer polynomials

$$(3.4)\qquad F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & b, b & uy, vy \\ q & c_1, \cdots, c_q & \\ 0 & \cdots & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (c_j)_n} C_n^b(t) y^n,$$

$|uy|<1$, $|vy|<1$, $u=t-(t^2-1)^{1/2}$, $v=t+(t^2-1)^{1/2}$.

Putting $b=\frac{1}{2}$ in (3.4), we get a generating relation for Legendre polynomials:

$$(3.5)\qquad F\left[\begin{array}{c|c|c} p & a_1, \cdots, a_p & \\ 1 & \frac{1}{2}, \frac{1}{2} & uy, vy \\ q & c_1, \cdots, c_q & \\ 0 & \cdots & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (c_j)_n} P_n(t) y^n,$$

$|uy|<1$, $|vy|<1$, $u=t-(t^2-1)^{1/2}$, $v=t+(t^2-1)^{1/2}$.

Throughout all our results, u and v stand for the same values as given above.

4. Particular cases.

I. In (3.2), putting $p=q=0$, $b'=\alpha$, $b=1+\alpha+\beta$, we get a known result [**6**, p. 120] due to Feldheim.

II. In (3.2), putting $p=q=1$, $b'=\alpha$, $b=1+\alpha+\beta$, we get

$$(4.1)\qquad F_1\left(a_1, 1+\alpha+\beta, -\alpha, c_1, \frac{(x-1)y}{2}, -y\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(c_1)_n} P_n^{(\alpha-n,\beta)}(x) y^n,$$

$$\left|\frac{(x-1)y}{2}\right| < 1, \quad |y| < 1.$$

Putting $c_1=1+\beta$ and applying (2.3), the result (4.1) degenerates to

$$(4.2)\qquad (1+y)^{-a_1}{}_2F_1\left(a_1, 1+\alpha+\beta; 1+\beta; \frac{y(1+x)}{2(1+y)}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(1+\beta)_n} P_n^{(\alpha-n,\beta)}(x)y^n.$$

III. In (3.2) putting $p=0$, $q=1$, $b'=\alpha$, $b=1+\alpha+\beta$, we get

$$(4.3)\qquad \Phi_2\left(1+\alpha+\beta, -\alpha, c_1, \frac{(x-1)y}{2}, -y\right) = \sum_{n=0}^{\infty} \frac{1}{(c_1)_n} P_n^{(\alpha-n,\beta)}(x)y^n.$$

Putting $c_1=1+\beta$ and applying (2.4), the result (4.3) reduces to

$$(4.4)\qquad e^{-y}{}_1F_1(1+\alpha+\beta; 1+\beta; y(1+x)/2) = \sum_{n=0}^{\infty} \frac{1}{(1+\beta)_n} P_n^{(\alpha-n,\beta)}(x)y^n.$$

IV. In (3.1), putting $p=q=0$, replacing x by x/b, making $b\to\infty$, and using (1.4), we get

$$(4.5)\qquad e^{-xy}(1+y)^{b'} = \sum_{n=0}^{\infty} L_n^{(b'-n)}(x)y^n, \qquad |y|<1,$$

which is relation (19) in [**5**, p. 189].

V. In (3.1), putting $p=q=1$, replacing x by x/b and making $b\to\infty$, and using (1.4), we get

$$(4.6)\qquad \Phi_1(a_1, -b', c_1, -y, -xy) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(c_1)_n} L_n^{(b'-n)}(x)y^n, \qquad |y|<1.$$

where Φ_1 is Horn's function [**4**, p. 225 (20)].

Putting $c_1=-b'$, the result (4.6) reduces to

$$(4.7)\qquad \sum_{n=0}^{\infty} \frac{(a_1)_n}{n!}(-y)^n{}_1F_1\left[\begin{matrix} a_1+n; \\ -b'+n; \end{matrix} -xy\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(-b')_n} L_n^{(b'-n)}(x)y^n.$$

VI. In (3.1) taking $p=0$, $q=1$, replacing x by x/b and making $b\to\infty$, and using (1.4), we get

$$(4.8)\qquad \Phi_3(-b', c_1, -y, -xy) = \sum_{n=0}^{\infty} \frac{1}{(c_1)_n} L_n^{(b'-n)}(x)y^n,$$

where Φ_3 is Horn's function [**4**, p. 225 (22)].

Putting $c_1=-b'$ in (4.8), we get

$$(4.9)\qquad \sum_{n=0}^{\infty} \frac{1}{n!}{}_0F_1\left[\begin{matrix} -; \\ -b'+n; \end{matrix} xy\right]y^n = \sum_{n=0}^{\infty} \frac{(b'-n)!}{b'!} L_n^{(b'-n)}(x)y^n.$$

VII. In (3.3), putting $p=q=0$, we get a well-known result [**3**, p. 88] due to Carlitz.

VIII. In (3.3), putting $p=q=1$, we get the Manocha's result [**10**, p. 457 (1.4)], which reduces to another result of his [**10**, p. 457 (1.5)] with the help of (2.3).

IX. In (3.3), putting $p=0$, $q=1$, we get

$$\Phi_2\left(-b, -b', c_1, \frac{(1-x)y}{2}, \frac{-(1+x)y}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{(c_1)_n} P_n^{(b'-n,b-n)}(x)y^n. \tag{4.10}$$

In (4.10), putting $c_1=-b-b'$ and applying (2.4), we get

$$\begin{aligned} &\exp\left[\frac{-(1+x)y}{2}\right] {}_1F_1[-b; -b-b'; y] \\ &\qquad = \sum_{n=0}^{\infty} \frac{1}{(-b-b')_n} P_n^{(b'-n,b-n)}(x)y^n, \end{aligned} \tag{4.11}$$

which is result (1.4) in [**9**] due to Manocha.

X. In (3.4), putting $q=p$, we get a known result [**7**, p. 28 (2)] due to Jain.

XI. In (3.5), putting $q=p$, we get a result [**7**, p. 29 (5)] due to Jain.

XII. In (3.4), putting $q=p=1$, we get another Jain's result [**7**, p. 29 (6)].

XIII. In (3.5), putting $q=p=1$, we obtain a result [**7**, p. 29 (7)] due to Jain.

XIV. In (3.4), taking $p=0$, $q=1$, we have

$$\Phi_2(b, b, c_1, uy, vy) = \sum_{n=0}^{\infty} \frac{1}{(c_1)_n} C_n^b(t)y^n. \tag{4.12}$$

XV. In (3.5), taking $p=0$, $q=1$, we obtain

$$\Phi_2(\tfrac{1}{2}, \tfrac{1}{2}, c_1, uy, vy) = \sum_{n=0}^{\infty} \frac{1}{(c_1)_n} P_n(t)y^n. \tag{4.13}$$

XVI. In (3.4), taking $p=q=0$, we get a known result [**11**, p. 276 (1)].

XVII. In (3.5), taking $p=q=0$, we get a known formula [**11**, p. 157 (1)].

XVIII. In (3.4), putting $p=0$, $q=1$, $c_1=2b$ and applying (2.4), we get

$$\exp(vy)\, {}_1F_1[b; 2b; (u-v)y] = \sum_{n=0}^{\infty} \frac{1}{(2b)_n} C_n^b(t)y^n, \tag{4.14}$$

which is a relation [**8**, p. 109 (2.1)] due to Jain.

On applying Kummer's second formula [**11**, p. 126 (9)] formula (4.14) assumes the form:

$$e^{yt}{}_0F_1\left[\begin{matrix} -; \\ b+\frac{1}{2}; \end{matrix} \frac{y^2(t^2-1)}{4}\right] = \sum_{n=0}^{\infty} \frac{1}{(2b)_n} C_n^b(t) y^n, \tag{4.15}$$

which is a known formula [**11**, p. 278 (7)].

Putting $t = \operatorname{Cos}\theta$ and using the definition of Bessel functions of first kind, the above result (4.15) changes to the well-known relation [**5**, p. 177 (30)].

XIX. In (3.5) putting $p=0$, $q=1$, $c_1=1$ and applying (2.4) we get

$$\exp(vy)\,{}_1F_1(\tfrac{1}{2}; 1; (u-v)y) = \sum_{n=0}^{\infty} \frac{P_n(t)}{n!} y^n. \tag{4.16}$$

Making the use of Kummer's second formula, the result (4.16) yields the known relation [**11**, p. 165 (4)] which, on using the definition of Bessel functions of first kind, assumes the form of the formula [**5**, p. 182 (40)].

XX. In (3.4), putting $q=p=1$, $c_1=2b$ and applying (2.3), we obtain

$$(1-vy)^{-a_1}{}_2F_1\left(a_1, b; 2b; \frac{(u-v)y}{1-vy}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(2b)_n} C_n^b(t) y^n. \tag{4.17}$$

Employing (2.2) in (4.17), we obtain a known result [**11**, p. 279 (8)], given below

$$(1-ty)^{-a_1}{}_2F_1\left[\begin{matrix} \frac{a_1}{2}, \frac{a_1+1}{2}; \\ b+\frac{1}{2}; \end{matrix} \frac{y^2(t^2-1)}{(1-ty)^2}\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(2b)_n} C_n^b(t) y^n. \tag{4.18}$$

Changing the result (4.18) into the form of an ultraspherical polynomial, with the help of [**11**, p. 277 (5)], we obtain a well-known result (18) in [**2**] due to Brafman.

XXI. In (3.5), putting $q=p=1$, $c_1=1$ and applying (2.3), we have

$$(1-vy)^{-a_1}{}_2F_1\left(a_1, \tfrac{1}{2}; 1; \frac{(u-v)y}{1-vy}\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{n!} P_n(t) y^n. \tag{4.19}$$

Using (2.2), the formula (4.19) yields the well-known formula (21) in [**2**] due to Brafman.

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