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SKELETA OF COMPLEXES WITH LOW MU_* PROJECTIVE DIMENSION[1]

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ABSTRACT. Let $MU_*(X)$ be the unitary bordism of a finite complex X. Let X^p be the p-skeleton of X. This note proves that certain properties of $MU_*(X)$ are shared by $MU_*(X^p)$ when the projective dimension of $MU_*(X)$ as a MU_* module is low (0, 1, or 2).

Introduction. Let X be a finite complex and let X^p be any skeleton of X. This note examines some algebraic properties which are inherited by the skeleton X^p from the complex X. Our work is motivated by the elementary example: if $H_*(X; Z)$ is free abelian, then $H_*(X^p; Z)$ is also free abelian.

$MU_*(\ )$ is the complex bordism homology theory associated to the Thom spectrum MU. We denote the projective dimension of $MU_*(X)$ as a $MU_*\equiv MU_*(\text{Point})$-module by $\text{hom dim}_{MU_*} MU_*(X)$. Connor and Smith prove: $\text{hom dim}_{MU_*} MU_*(X)=0$ if and only if $H_*(X; Z)$ is free abelian [**3**, 3.10]. Our elementary example becomes the $n=0$ version of our first theorem.

THEOREM 1. *Let X be a finite complex. If $n=0$, 1, or 2, then* $\text{hom dim}_{MU_*} MU_*(X)\leqq n$ *if and only if* $\text{hom dim}_{MU_*} MU_*(X^p)\leqq n$ *for every skeleton X^p of X.*

The "if" part of the theorem is trivial. The $n=1$ version is a folk theorem; we shall sketch a proof for completeness.

$k_*(\ )$ is the connective k-theory; it is the homology theory derived from the connected unitary spectrum ***bu*** ([**1**], [**8**]). We prove:

THEOREM 2. *Let X be a finite complex. The following four conditions are equivalent.*

(i) *$k_*(X)$ is free abelian.*
(ii) *$k_*(X^p)$ is free abelian for every skeleton X^p of X.*
(iii) *$MU_*(X^p)$ is free abelian for every skeleton X^p of X.*
(iv) *$MU_*(X)$ is free abelian and* $\text{hom dim}_{MU_*} MU_*(X)\leqq 1$.

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As there exist finite complexes X with $MU_*(X)$ free abelian and with hom $\dim_{MU_*} MU_*(X)$ arbitrarily high, we see that $MU_*(X)$ free abelian does not imply $MU_*(X^p)$ is also.

Postnikov fibres of homology theories. Let $M_*(\ )$ be a homology theory and q an integer. We follow Dold [4] in defining the qth Postnikov fibre of $M_*(\ )$ to be the homology theory $M(q)_*(\ )$ with groups $M(q)_{p+q}(X)$ for a finite complex X given by

$$M(q)_{p+q}(X) = \text{Image}\{M_{p+q}(X^{p-1}) \to M_{p+q}(X^p)\}.$$

Recall the skeletal filtration exact couple for the Atiyah-Hirzebruch-Dold spectral sequence for $M_*(X)$ has $D^1_{p,q}=M_{p+q}(X^p)$ and $E^1_{p,q}=M_{p+q}(X^p, X^{p-1})$. So

$$M(q)_{p+q}(X) = \text{Image}\{D^1_{p-1,q+1} \to D^1_{p,q}\}$$

which is $D^2_{p,q}$ in the derived exact couple. There is a natural homomorphism $M(q)_{p+q}(X) \to M(q-1)_{p+q}(X)$ which is $i^2_{p,q}: D^2_{p,q} \to D^2_{p+1,q-1}$ in the derived exact couple [7, §XI–5]. The converse of the following folk lemma also holds, but we shall not need it.

Lemma 3. *$M(q)_{p+q}(X) \to M(q-1)_{p+q}(X)$ is a monomorphism modulo torsion. When it is monic, the Atiyah-Hirzebruch-Dold spectral sequence for $M_*(X)$ collapses.*

Proof. The first statement is seen by tensoring the derived exact couple with $\boldsymbol{Q}$, the rationals. For the second statement, note that if all the $i^2_{p,q}$'s are monic in the derived exact couple, then all the $k^2_{p,q}$'s are zero. Since each differential involves a $k^2_{p,q}$, the spectral sequence collapses. Q.E.D.

We shall work with three homology theories: integral homology, connective k-theory, and complex bordism. All three satisfy the hypothesis of Lemma 4.

Lemma 4. *Let $M_*(\ )$ be a homology theory such that $M_{2n+1} \equiv M_{2n+1}(Point)=0$ for each integer n. Then*

$$M(2r+1)_*(X) \cong M(2r)_*(X)$$

for any finite complex X.

Proof. This follows immediately from definitions and the fact that $M_{2r+1+s}(X^s, X^{s-1})=0$. Q.E.D.

Let $f: A_*(\ ) \to B_*(\ )$ be a natural transformation of two homology theories. The inclusion of skeleta, $i: X^{p-1} \to X^p$, of a finite complex X

induces the commuting diagram:

$$\begin{array}{ccc} A_{p+q}(X^{p-1}) & \xrightarrow{A_*(i)} & A_{p+q}(X^p) \\ \downarrow f_1 & & \downarrow f_2 \\ B_{p+q}(X^{p-1}) & \xrightarrow{B_*(i)} & B_{p+q}(X^p) \end{array}$$

we may define

$$f(q)\colon A(q)_{p+q}(X) = \text{Image } A_*(i) \to B(q)_{p+q}(X) = \text{Image } B_*(i)$$

by

$$f(q)(A_*(i)(y)) = f_2A_*(i)(y) = B_*(i)f_1(y) \quad \text{for any } y \in A_{p+q}(X^{p-1}).$$

In particular: if spectra $\boldsymbol{A}$ and $\boldsymbol{B}$ represent the homology theories $A_*(\ )$ and $B_*(\ )$, a degree 0 morphism of spectra $\boldsymbol{f}\colon \boldsymbol{A}\to\boldsymbol{B}$ induces natural transformations $f\colon A_*(\ )\to B_*(\ )$ and $f(q)\colon A(q)_*(\ )\to B(q)_*(\ )$. Let

$$\boldsymbol{A} \xrightarrow{\boldsymbol{f}} \boldsymbol{B} \xrightarrow{\boldsymbol{q}} \boldsymbol{C} \xrightarrow{\boldsymbol{h}} \boldsymbol{A}$$

be an exact triangle of spectra where $\boldsymbol{f}$ and $\boldsymbol{g}$ are of degree 0 and $\boldsymbol{h}$ is of degree -1 [**10**, §XI]. This induces a long exact sequence of homology theories:

$$\cdots \longrightarrow A_n(\ ) \xrightarrow{f} B_n(\ ) \xrightarrow{g} C_n(\ ) \xrightarrow{h} A_{n-1}(\ ) \longrightarrow \cdots.$$

We start to ask whether $f(q), g(q), \cdots$ are in a long exact sequence, but we recall that $h(q)$ has not been (and may not be) defined.

LEMMA 5. *If* $\boldsymbol{A}\xrightarrow{\boldsymbol{f}}\boldsymbol{B}\xrightarrow{\boldsymbol{q}}\boldsymbol{C}\xrightarrow{\boldsymbol{h}}\boldsymbol{A}$ *is an exact triangle of spectra as above such that the homology theories* $A_*(\ )$, $B_*(\ )$, *and* $C_*(\ )$ *satisfy the hypothesis of Lemma* 4, *then there is a long exact sequence of homology theories*:

$$\cdots \longrightarrow A(2r)_n(\ ) \xrightarrow{f(2r)} B(2r)_n(\ ) \xrightarrow{g(2r)} C(2r)_n(\ )$$
$$\xrightarrow{h(2r)} A(2r)_{n-1}(\ ) \longrightarrow \cdots.$$

PROOF. For a finite complex X, let $i\colon X^{n-2r-1}\to X^{n-2r}$ and $j\colon X^{n-2r-2}\to X^{n-2r-1}$ be inclusions of skeleta. These induce the exact rows in the diagram below.

$$\begin{tikzcd} & C_{n+1}(X^{n-2r}, X^{n-2r-1}) \arrow[d, phantom, "="] & \\ & 0 \arrow[r] & C_n(X^{n-2r-1}) \arrow[r, "C_n(i)"] \arrow[d, "1"] & C_n(X^{n-2r}) \\ C_n(X^{n-2r-2}) \arrow[rr, "C_n(j)"] \arrow[d, "h_1"] & & C_n(X^{n-2r-1}) \arrow[r] \arrow[d, "h_2"] & C_n(X^{n-2r-1}, X^{n-2r-2}) = 0 \\ A_{n-1}(X^{n-2r-2}) \arrow[rr, "A_{n-1}(j)"] & & A_{n-1}(X^{n-2r-1}). & \end{tikzcd}$$

Define

$$h(2r)\colon C(2r)_n(X) = \text{Image } C_n(i) \to A(2r)_{n-1}(X) = \text{Image } A_{n-1}(j)$$

by

$$h(2r)(C_n(i)(y)) = h_2(y) = A_{n-1}(j) \circ h_1 \circ C_n(j)^{-1}(y)$$

for any $y \in C_n(X^{n-2r-1})$.

Verification of exactness is routine. Q.E.D.

LEMMA 6. *Let X be a finite complex. There is a finite complex Y and a map $g\colon Y \to \Sigma^s X$ ($\Sigma^s X$ is the s-fold suspension of X) such that:*

(i) *$H_*(Y)$ is free abelian;*

(ii-1) *$g_*\colon H(2r)_*(Y) \to H(2r)_*(\Sigma^s X)$ is epic if* hom $\dim_{MU_*} MU_*(X) \leqq 1$;

or

(ii-2) *$g_*\colon k(2r)_*(Y) \to k(2r)_*(\Sigma^s X)$ is epic if* hom $\dim_{MU_*} MU_*(X) \leqq 2$.

PROOF. $H(2r)_*(\ ) = H_*(\ ; Z)$ for r negative and $H(2r)_*(\ )$ is the zero homology theory for r nonnegative. From Bott periodicity, there is the isomorphism $k(2r)_n(\ ) \cong k_{n-2r}(\ )$ **[1]**. The lemma follows from the existence of MU-resolutions (**[3]**, 2.4 and 3.11], **[6]**). Q.E.D.

PROOF OF THEOREM 1. *The $n=2$ case.* We assume X is a finite complex with hom $\dim_{MU_*} MU_*(X) \leqq 2$.

The Thom homomorphism $\zeta\colon MU_*(\ ) \to k_*(\ )$ is induced by a degree 0 morphism of spectra $\boldsymbol{\zeta}\colon \boldsymbol{MU} \to \boldsymbol{bu}$. By **[10]**, we may construct an exact triangle of spectra

$$\boldsymbol{A} \xrightarrow{\boldsymbol{f}} \boldsymbol{MU} \xrightarrow{\boldsymbol{\zeta}} \boldsymbol{bu} \xrightarrow{\boldsymbol{h}} \boldsymbol{A}$$

with degree($\boldsymbol{f}$)$=0$ and degree($\boldsymbol{h}$)$=-1$. For any finite complex X, we have the long exact sequence

$$\begin{tikzcd} \arrow[r] & A_*(X) \arrow[r, "f"] & MU_*(X) \arrow[r, "\zeta"] & k_*(X) \arrow[lll, rounded corners, to path={-- ([yshift=-4ex]\tikztostart.south) -| node[near start, below]{h} (\tikztotarget)}] \end{tikzcd}$$

(degree(h)$=-1$). By **[5]**, hom $\dim_{MU_*} MU_*(X) \leqq 2$ if and only if ζ is epic. We conclude: hom $\dim_{MU_*} MU_*(X) \leqq 2$ if and only if f is monic. If $H_*(Y; Z)$ is free abelian and if X is Y or Y^p, then f is monic. In particular if X is a one point space, this proves that $A_*(\ )$ satisfies the hypothesis of Lemma 4.

We claim: hom $\dim_{MU_*} MU_*(X^p) \leqq 2$ if and only if $f(q)\colon A(q)_{p+q}(X) \to MU(q)_{p+q}(X)$ is monic for all integers p and q. From the exact sequences for the pair (X^p, X^{p-1}), we have exact rows in the diagram below.

$$\begin{tikzcd} 0 \arrow[r] & A(q)_{p+q}(X) \arrow[r] \arrow[d, "f(q)"] & A_{p+q}(X^p) \arrow[r] \arrow[d, "f_1"] & A_{p+q}(X^p, X^{p-1}) \arrow[d, "f_2"] \\ 0 \arrow[r] & MU(q)_{p+q}(X) \arrow[r] & MU_{p+q}(X^p) \arrow[r] & MU_{p+q}(X^p, X^{p-1}). \end{tikzcd}$$

Since $H_*(X^p, X^{p-1}; Z)$ is free abelian, f_2 is monic. By the "5" Lemma, f_1 is monic if and only if $f(q)$ is monic. This proves the claim.

To prove the proposition, we pick $g\colon Y \to \Sigma^s X$ as in Lemma 6. Since $H^*(Y; Z)$ is free abelian, $f(2r)'\colon A(2r)_* Y \to MU(2r)_*(Y)$ is monic and the top row in our diagram below is short exact. The bottom sequence is exact by Lemma 5.

$$\begin{tikzcd}
0 \arrow[r] & A(2r)_*(Y) \arrow[r, "f(2r)'"] \arrow[d, "g_1"] & MU(2r)_*(Y) \arrow[r, "\zeta(2r)'"] \arrow[d, "g_2"] & k(2r)_*(Y) \arrow[r] \arrow[d, "g_3"] & 0 \\
 & A(2r)_*(\Sigma^s X) \arrow[r, "f(2r)"] & MU(2r)_*(\Sigma^s X) \arrow[r, "\zeta(2r)"] & k(2r)_*(\Sigma^s X) \arrow[ll, bend left, "h(2r)"] &
\end{tikzcd}$$

g_3 is epic by Lemma 6. So $\zeta(2r)$ is epic and $f(2r)$ is monic. By desuspending (and by Lemma 4 for odd q), we have $f(q)\colon A(q)_{p+q}(X) \to MU(q)_{p+q}(X)$ is monic for all p and q. By our claim this proves hom $\dim_{MU_*} MU_*(X^p) \leqq 2$.

The $n=1$ case. There is a degree 0 morphism of spectra $\mu\colon \boldsymbol{MU} \to \boldsymbol{K}(\boldsymbol{Z})$ ($\boldsymbol{K}(\boldsymbol{Z})$ is the Eilenberg-Mac Lane spectrum) inducing the Thom homomorphism $\mu\colon MU_*(\ ) \to H_*(\ ; Z)$. hom $\dim_{MU_*} MU_*(X) \leqq 1$ if and only if μ is epic for X a finite complex [**3**, 3.11]. Now one can mimic the proof of the $n=2$ case. Q.E.D.

Proof of Theorem 2. (i) *implies* (ii). Since $k(2r+1)_n(X) \simeq k(2r)_n(X) \simeq k_{n-2r}(X)$ (Lemma 4 and the proof of Lemma 6), we have that $k(q)_{p+q}(X)$ is free abelian for all p and q. The exact sequence of the pair (X^p, X^{p-1}) induces the exact sequence:

$$0 \to k(q)_{p+q}(X) \to k_{p+q}(X^p) \to k_{p+q}(X^p, X^{p-1}).$$

$k_{p+q}(X^p, X^{p-1})$ and $k(q)_{p+q}(X)$ are free abelian; so $k_{p+q}(X^p)$ is also.

(ii) *implies* (iii). Since $k_*(X^p)$ is free abelian, hom $\dim_{MU_*} MU_*(X^p) \leqq 1$ [**5**]. So the Stong-Hattori homomorphism

$$sh\colon MU_*(X^p) \to k_*(\boldsymbol{MU} \wedge X^p) \simeq k_*(\boldsymbol{MU}) \otimes_{Z[t]} k_*(X^p)$$

is monic [**9**]. $k_*(\boldsymbol{MU})$ is a free $Z[t]$-module; so $k_*(\boldsymbol{MU} \wedge X^p)$ is free abelian.

(iii) *implies* (iv). $MU(q)_{p+q}(X)$ is a subgroup of the free abelian $MU_{p+q}(X^p)$; so the monomorphism modulo torsion $MU(q)_{p+q}(X) \to MU(q-1)_{p+q}(X)$ is monic. By Lemma 3, the Atiyah-Hirzebruch-Dold spectral sequence for $MU_*(X)$ collapses. By [**3**, 3.11], this implies hom $\dim_{MU_*} MU_*(X) \leqq 1$.

(iv) *implies* (i). Since hom $\dim_{MU^*} MU_*(X) \leqq 1$, there is a monomorphism $m_t\colon k_{2n}(X) \to k_{2n+2}(X)$ [**5**]. We may consider $k_{2n}(X)$ as a subgroup of $K_0(X) = k_{2n+2r}(X)$ for r large. But $K_0(X)$ is a direct summand of $MU_0(X)$

[2, §9]; so $k_{2n}(X)$ is free abelian. Repeat the argument to get $k_{2n-1}(X) \simeq \tilde{k}_{2n}(\Sigma X^+)$ free abelian. Q.E.D.

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