H-STRUCTURES ON PRODUCTS

ARTHUR H. COPELAND, JR.

Abstract. This note outlines a technique for determining the set of homotopy classes of H-structures on a product of H-spaces. The technique is then applied to computing the number of H-structures on SO(4).

1. Introduction. This note outlines a technique for determining the set of homotopy classes of H-structures on a product $X = X_1 \times X_2 \times \cdots \times X_n$ of H-spaces. This technique is then applied to computing the number of H-structures on SO(4) = SO(3) × S^3.

The technique may be summarized in the following statements.

1. The set of homotopy classes of H-structures on X is in one-to-one correspondence with the homotopy set $[X \wedge X; X]$.

2. There are subcomplexes $* = L_0 \subset L_1 \subset \cdots \subset L_{2n} = X \wedge X$ such that the sequences $0 \to [L_r/L_{r-1}; X] \to [L_r; X] \to [L_{r-1}; X] \to 0$ are exact for $r = 1, 2, \ldots, 2n$.

3. $[L_r/L_{r-1}; X]$ is the direct product of the loops $[X_i \wedge \cdots \wedge X_p \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}; X]$ with $p, q \geq 1$ and $p + q = r$.

4. For any pointed space L, $[L; X]$ is the direct product of the loops $[L; X_k]$ for $k = 1, 2, \ldots, n$.

Statement (4) is completely routine and (1) is essentially Theorem 5.5A of [1]. Statements (2) and (3) are established in §2. Throughout this note all spaces should be assumed to be CW-complexes of finite type.

As a corollary to the statements (1)–(4) above, one sees that the number $\# [X \wedge X; X]$ of products on X is the product of the numbers $\# [X \wedge X; X] = \# [X_1 \wedge \cdots \wedge X_p \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}; X_k]$.

2. Main results. We assume that each of the spaces $X_k$ ($k = 1, 2, \ldots, n$) is a path-connected, pointed CW-complex of finite type and has at least one H-structure. As usual, * will be used ambiguously to denote basepoints. Let $X = X_1 \times \cdots \times X_n$ and let $\phi : X \times X \to X \wedge X$ be the usual identification map. Suppose $p, q$ are integers with $1 \leq p, q \leq n$ and that $a = (i_1, \cdots, i_p, j_1, \cdots, j_q)$ is a $(p+q)$-tuple of integers with $1 \leq i_1 < \cdots < i_p \leq n$.  

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and \(1 \leq j_1 \leq j_2 \leq \cdots \leq j_n \leq n\). Such an \(\alpha\) will be called admissible. Define \(\#\alpha = p + q\) and let \(K_\alpha \subset X \wedge X\) be the set of points \(\phi(x_1, \cdots, x_n, x'_1, \cdots, x'_n)\) with \(x_i = x'_i = *\) except when \(i = i_1, \cdots, i_p\) or \(j = j_1, \cdots, j_q\). Let \(L_r = \bigcup \{K_\alpha \mid \#\alpha = r\}\) for \(r = 2, 3, \cdots, 2n\) and let \(L_1 = * = \phi(X \wedge X)\). Note that \(L_1 \subset L_2 \subset \cdots \subset L_{2n} = X \wedge X\). Let \(M_\alpha = X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}\).

2.1 Lemma (James). If \(Y_1, \cdots, Y_m\) are subcomplexes of \(Y\) and if \(r_i : Y \to Y_i\) are retractions with \(r_i(Y_j) \subset Y_j\) for all \(i\) and \(j\), then \(Y_1 \cup \cdots \cup Y_m\) is retractile in \(Y\) [3, Lemma 3.1].

2.2 Lemma. If \(r \leq s\), then \(L_r\) is retractile in \(L_s\).

Proof. In view of \([8, Proposition (3.7)]\) it suffices to show each \(L_r\) is retractile in \(X \wedge X\). But the projections

\[
X \times X \to X_{i_1} \times \cdots \times X_{i_p} \times X_{j_1} \times \cdots \times X_{j_q}
\]

induce retractions \(X \wedge X \to K_\alpha\) satisfying the hypothesis of James' Lemma.

2.3 Theorem. The sequences

\[
0 \to [L_r/L_{r-1}; X] \to [L_r; X] \to [L_{r-1}; X] \to 0
\]

are exact for \(r = 2, 3, \cdots, 2n\).

This is an immediate consequence of O'Neill's work \([8, Theorems (3.3)\) and (3.4)]\). Note that if \(X\) is the space of loops on an \(H\)-space, then a result of Eckmann and Hilton \([2, Theorem 2.2]\) implies that these sequences split.

2.4 Theorem. \([L_r/L_{r-1}; X]\) is the direct product of the loops \([M_\alpha; X]\) with \(\#\alpha = r\).

Proof. It suffices to establish that \(L_r/L_{r-1} = \bigvee \{M_\alpha \mid \#\alpha = r\}\). But if \(\alpha \neq \beta\) are admissible \(r\)-triples, then \(K_\alpha \cap K_\beta \subset L_{r-1}\). The result now follows from the observation that \(K_\alpha / (K_\alpha \cap L_{r-1}) = M_\alpha\).

3. Application to \(SO(4)\).

3.1 Theorem. The space \(X = SO(4)\) has \(2^{64} \cdot 3^{16}\) homotopy classes of \(H\)-structures.

In the course of the proof it will be convenient to refer to the following table of values of \([\Sigma^k P^2; S^3]\).

<table>
<thead>
<tr>
<th>(k)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\Sigma^k P^2; S^3])</td>
<td>(2)</td>
<td>(4)</td>
<td>(2) + (2)</td>
<td>(2) + (2)</td>
<td>(4)</td>
<td>(2)</td>
<td>(1)</td>
<td>(2)</td>
</tr>
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</table>
The symbol \((m)\) denotes the abelian group of integers modulo \(m\), and \(\oplus\) denotes direct sums. The entries are readily deduced from the Appendix to Chapter 12 of Hilton's book [4].

**Proof of 3.1.** We have \(X_1 = P^3 = SO(3), X_2 = S^3\). Note that since \(X \wedge X\) is simply-connected, \([X \wedge X; S^3]\) is isomorphic to \([X \wedge X; X_1]\) under the 2-fold covering projection \(S^3 \rightarrow P^3\). Let

\[
\begin{align*}
a &= \# [X_1 \wedge X_2 \wedge X_1 \wedge X_2; S^3] = \# [S^6 \wedge P^3 \wedge P^3; S^3], \\
b_1 &= \# [X_1 \wedge X_1 \wedge X_1; S^3] = \# [S^3 \wedge P^3 \wedge P^3; S^3], \\
b_2 &= \# [X_1 \wedge X_2 \wedge X_2; S^3] = \# [S^6 \wedge P^3; S^3], \\
c_1 &= \# [X_1 \wedge X_1; S^3], \\
c_2 &= \# [X_1 \wedge X_2; S^3] = \# [S^3 \wedge P^3; S^3] \quad \text{and} \\
c_3 &= \# [X_2 \wedge X_2; S^3] = \# [S^6; S^3];
\end{align*}
\]

then \([X \wedge X; X] = a^3 b_1^3 c_2^3 c_3^3\). The evaluation of these numbers makes repeated use of Naylor's theorem that \(S^3 \wedge P^3 = (S^3 \wedge P^3) \vee S^6\) [6]. In addition, we will use Table 3.2 several times in each calculation.

**The value of \(a\).**

\[
S^6 \wedge P^3 \wedge P^3 = (S^3 \wedge P^3) \wedge (S^3 \wedge P^3) \wedge S^6.
\]

Thus

\[
a = \# [S^6 \wedge P^3 \wedge P^3; S^3] \cdot \# [S^6 \wedge P^3; S^3] \cdot \# [S^6; S^3].
\]

The cofibration sequence of spaces

\[
S^1 \rightarrow S^1 \rightarrow P^2 \rightarrow S^2 \rightarrow S^2
\]

in which \(f\) ambiguously denotes maps of degree 2, induces the cofibration sequence

\[
\Sigma^7 P^2 \rightarrow \Sigma^7 P^2 \rightarrow S^6 \wedge P^2 \wedge P^2 \rightarrow \Sigma^8 P^2 \rightarrow \Sigma^8 P^2
\]

and hence the exact sequence of groups

\[
\Sigma^7 P^2; S^3 \quad \Sigma^6 P^2; P^2; S^3 \quad \Sigma^8 P^2; S^3
\]

The two outer groups are repetitions and have been omitted from the line above. Since \([\Sigma^7 P^2; S^3] = Z_2\) and \([\Sigma^8 P^2; S^3] = 0\), both homomorphisms \(f^*\) are trivial, and \(\# [S^8 \wedge P^2 \wedge P^2; S^3] = 2\).

On the other hand, \(\# [\Sigma^9 P^2; S^3] = 2\) and \(\# [S^{12}; S^3] = 4\). Thus \(a = 2^5\).

**The value of \(b_1\).**

\[
S^5 \wedge P^3 \wedge P^3 = ((S^3 \wedge P^3) \wedge P^3) \wedge (P^2 \wedge S^3 \wedge P^3) \vee S^6 \wedge P^3 = (S^3 \wedge P^2 \wedge P^3) \vee (\Sigma^6 P^2) \vee (\Sigma^9 P^2) \vee S^9.
\]
Thus $b_1 = \#[S^3 \wedge P^2 \wedge P^2; S^3]$. As before, we have an exact sequence

$$\xymatrix{ [\Sigma^4 P^2; S^3] \ar[r]^f & [S^3 \wedge P^2 \wedge P^2; S^3] \ar[r] & [\Sigma^5 P^2; S^3] }$$

But $[\Sigma^4 P^2; S^3] \approx [\Sigma^4 P^2; S^3] \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $f^*$ is trivial on each of these. This shows $\#(S^3 \wedge P^2 \wedge P^2; S^3) = 2^4$.

On the other hand $\#(\Sigma^4 P^2; S^3) = 2^8$ and $\#\pi_6(S^3) = 3$, whence $b_1 = 2^8 \cdot 3$.

The values of $b_2$, $c_1$, $c_2$ and $c_3$.

- $b_2 = \#([S^6 \wedge P^3; S^3]) = \#([S^6 \wedge P^2; S^3] \cdot [S^6; S^3]) = 2^8 \cdot 3$;
- $c_1 = \#([P^3 \wedge P^3; S^3]) = 2^8 \cdot 3$ [4];
- $c_2 = \#([S^3 \wedge P^3; S^3]) = \#([S^3 \wedge P^2; S^3] \cdot [S^6; S^3]) = 4 \cdot 12 = 2^4 \cdot 3$;
- $c_3 = \#\pi_6(S^3) = 2^8 \cdot 3$.

Thus $\#(X \wedge X; X) = 2^{30} \cdot 3^{16}$.

A similar calculation would show that $S^3 \times S^3$ has $2^{20} \cdot 3^{16}$ homotopy classes of $H$-structures. This result was obtained earlier by Norman [7] using Loibel's Theorem [5].

**BIBLIOGRAPHY**


Department of Mathematics, University of New Hampshire, Durham, New Hampshire 03824