1. Introduction. This note outlines a technique for determining the set of homotopy classes of $H$-structures on a product of $H$-spaces. The technique is then applied to computing the number of $H$-structures on $SO(4)$.

The technique may be summarized in the following statements.

1. The set of homotopy classes of $H$-structures on $A=A_1 \times A_2 \times \cdots \times A_n$ is in one-to-one correspondence with the homotopy set $[A \wedge A; A]$.

2. There are subcomplexes $* = L_0 \subset L_1 \subset \cdots \subset L_{2n} = X \wedge X$ such that the sequences $0 \to [L_r/L_{r-1}; X] \to [L_r; X] \to [L_{r-1}; X] \to 0$ are exact for $r = 1, 2, \cdots, 2n$.

3. $[L_r/L_{r-1}; X]$ is the direct product of the loops $[X_i \wedge \cdots \wedge X_p \wedge X_{i_1} \wedge \cdots \wedge X_{i_q}; X]$ with $p, q \geq 1$ and $p+q=r$.

4. For any pointed space $L$, $[L; X]$ is the direct product of the loops $[L; X_k]$ for $k = 1, 2, \cdots, n$.

Statement (4) is completely routine and (1) is essentially Theorem 5.5A of [1]. Statements (2) and (3) are established in §2. Throughout this note all spaces should be assumed to be CW-complexes of finite type.

As a corollary to the statements (1)–(4) above, one sees that the number $\#[A \wedge A; A]$ of products on $A$ is the product of the numbers $\#[X_i \wedge \cdots \wedge X_p \wedge X_{i_1} \wedge \cdots \wedge X_{i_q}; X_k]$.

2. Main results. We assume that each of the spaces $X_k$ $(k = 1, 2, \cdots, n)$ is a path-connected, pointed CW-complex of finite type and has at least one $H$-structure. As usual, $*$ will be used ambiguously to denote basepoints. Let $X = X_1 \times \cdots \times X_n$ and let $\phi: X \times X \to X \wedge X$ be the usual identification map. Suppose $p, q$ are integers with $1 \leq p, q \leq n$ and that $\alpha = (i_1, \cdots, i_p, j_1, \cdots, j_q)$ is a $(p+q)$-tuple of integers with $1 \leq i_1 < i_2 < \cdots < i_p \leq n$.
and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$. Such an $\alpha$ will be called \textit{admissible}. Define $\# \alpha = p + q$ and let $K_\alpha \subset X \wedge X$ be the set of points $\phi(x_1, \cdots, x_n, x'_1, \cdots, x'_n)$ with $x_i = x'_i = *$ except when $i = i_1, \cdots, i_p$ or $j = j_1, \cdots, j_q$. Let $L_r = \bigcup \{K_\alpha | \# \alpha = r\}$ for $r = 2, 3, \cdots, 2n$ and let $L_1 = * = \phi(X \vee X)$. Note that $L_1 \subset L_2 \subset \cdots \subset L_{2n} = X \wedge X$. Let $M_\alpha = X_{i_1} \wedge \cdots \wedge X_{i_p} \wedge X_{j_1} \wedge \cdots \wedge X_{j_q}$.

2.1 \textbf{Lemma (James)}. \textit{If $Y_1, \cdots, Y_m$ are subcomplexes of $Y$ and if $r_i : Y \to Y_j$ are retractions with $r_i(Y_j) \subset Y_j$ for all $i$ and $j$, then $Y_1 \cup \cdots \cup Y_m$ is retractile in $Y$ [3, Lemma 3.1].}

2.2 \textbf{Lemma}. \textit{If $r \leq s$, then $L_r$ is retractile in $L_s$.}

\textbf{Proof.} In view of [8, Proposition (3.7)] it suffices to show each $L_r$ is retractile in $X \wedge X$. But the projections

$$X \times X \to X_{i_1} \times \cdots \times X_{i_p} \times X_{j_1} \times \cdots \times X_{j_q}$$

induce retractions $X \wedge X \to K_\alpha$ satisfying the hypothesis of James’ Lemma.

2.3 \textbf{Theorem}. \textit{The sequences}

$$0 \to [L_r/L_{r-1}; X] \to [L_r; X] \to [L_{r-1}; X] \to 0$$

\textit{are exact for $r = 2, 3, \cdots, 2n$.}

This is an immediate consequence of O’Neill’s work [8, Theorems (3.3) and (3.4)]. Note that if $X$ is the space of loops on an $H$-space, then a result of Eckmann and Hilton [2, Theorem 2.2] implies that these sequences split.

2.4 \textbf{Theorem}. \textit{$[L_r/L_{r-1}; X]$ is the direct product of the loops $[M_\alpha; X]$ with $\# \alpha = r$.}

\textbf{Proof.} It suffices to establish that $L_r/L_{r-1} = V\{M_\alpha | \# \alpha = r\}$. But if $\alpha \neq \beta$ are admissible $r$-triples, then $K_\alpha \cap K_\beta \subset L_{r-1}$. The result now follows from the observation that $K_\alpha/(K_\alpha \cap L_{r-1}) = M_\alpha$.

3. \textbf{Application to SO(4)}.

3.1 \textbf{Theorem}. \textit{The space $X = SO(4)$ has $2^{68} \cdot 3^{16}$ homotopy classes of $H$-structures.}

In the course of the proof it will be convenient to refer to the following table of values of $[\Sigma^k \mathbb{P}^2; S^3]$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
$k$ & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
$[\Sigma^k \mathbb{P}^2; S^3]$ & (2) & (4) & (2) & (2) & (2) & (2) & (2) & (1) \\
\hline
\end{tabular}
\end{table}
The symbol \((m)\) denotes the abelian group of integers modulo \(m\), and \(+\) denotes direct sums. The entries are readily deduced from the Appendix to Chapter 12 of Hilton’s book [4].

Proof of 3.1. We have \(X_1 = P^3 = SO(3)\), \(X_2 = S^3\). Note that since \(X \land X\) is simply-connected, \([X \land X; S^3]\) is isomorphic to \([X \land X; X_1]\) under the 2-fold covering projection \(S^3 \rightarrow P^3\). Let

\[
\begin{align*}
a &= \# [X_1 \land X_2 \land X_1 \land X_2; S^3] = \# [S^6 \land P^3 \land P^3; S^3], \\
b_1 &= \# [X_1 \land X_1 \land X_3; S^3] = \# [S^3 \land P^3 \land P^3; S^3], \\
b_2 &= \# [X_1 \land X_2 \land X_2; S^3] = \# [S^6 \land P^3; S^3], \\
c_1 &= \# [X_1 \land X_1; S^3], \\
c_2 &= \# [X_1 \land X_2; S^3] = \# [S^3 \land P^3; S^3] \quad \text{and} \\
c_3 &= \# [X_2 \land X_2; S^3] = \# [S^6; S^3];
\end{align*}
\]

then \([X \land X; X] = a^2 b_1 c_1 c_2 c_3\). The evaluation of these numbers makes repeated use of Naylor’s theorem that \(S^3 \land P^3 = (S^3 \land P^2) \lor S^9\) [6]. In addition, we will use Table 3.2 several times in each calculation.

The value of \(a\).

\[
S^6 \land P^3 \land P^3 = ((S^3 \land P^2) \lor S^9) \land ((S^3 \land P^2) \lor S^9).
\]

Thus

\[
a = \# [S^6 \land P^2 \land P^2; S^3] \cdot (\# [S^9 \land P^2; S^3])^2 \cdot \# [S^{12}; S^3].
\]

The cofibration sequence of spaces

\[
S^1 \rightarrow S^1 \rightarrow P^2 \rightarrow S^2 \rightarrow S^2
\]

in which \(f\) ambiguously denotes maps of degree 2, induces the cofibration sequence

\[
\Sigma^3 P^2 \rightarrow \Sigma^3 P^2 \rightarrow S^6 \land P^2 \land P^2 \rightarrow \Sigma^9 P^2 \rightarrow \Sigma^9 P^2
\]

and hence the exact sequence of groups

\[
\xymatrix{ \Sigma^3 P^2; S^3 \ar[r]^{f^*} & [S^6 \land P^2 \land P^2; S^3] \ar[l] \ar[r] & [S^8 P^2; S^3] \ar[l] \ar[r] & [\Sigma^8 P^2; S^3] \ar[l] }.
\]

The two outer groups are repetitions and have been omitted from the line above. Since \([\Sigma^3 P^2; S^3] = Z_2\) and \([\Sigma^8 P^2; S^3] = 0\), both homomorphisms \(f^*\) are trivial, and \(# [S^8 \land P^2 \land P^2; S^3] = 2\).

On the other hand, \(# [\Sigma^3 P^2; S^3] = 2\) and \(# [S^{12}; S^3] = 4\). Thus \(a = 2^5\).

The value of \(b_1\).

\[
S^3 \land P^3 \land P^3 = ((S^3 \land P^2) \land S^6) \land P^3 = (P^2 \land S^3 \land P^3) \lor S^6 \land P^3
\]

\[
= (S^3 \land P^2 \land P^2) \lor (S^6 P^2) \lor (S^6 P^2) \lor S^9.
\]
Thus \( b_1 = \#([S^3 \wedge P^2 \wedge P^2; S^3]) \cdot \#([S^6 P^2; S^3]) \cdot \#([S^9; S^3]). \) As before, we have an exact sequence

\[
\begin{array}{ccc}
\Sigma^4 P^2; S^3 & \xrightarrow{f^*} & [S^3 \wedge P^2 \wedge P^2; S^3] & \xrightarrow{\partial} & [\Sigma^5 P^2; S^3]
\end{array}
\]

But \( [\Sigma^4 P^2; S^3] \approx [\Sigma^6 P^2; S^3] \approx \mathbb{Z}_9 + \mathbb{Z}_9, \) and \( f^* \) is trivial on each of these. This shows \( \#([S^3 \wedge P^2 \wedge P^2; S^3]) = 2^4. \)

On the other hand \( \#([\Sigma^6 P^2; S^3]) = 2^9 \) and \( \# \pi_6 (S^3) = 3, \) whence \( b_1 = 2^8 \cdot 3. \)

The values of \( b_2, c_1, c_2 \) and \( c_3, \)

\[
\begin{align*}
b_2 &= \#([S^6 \wedge P^3; S^3]) = \#([S^6 \wedge P^2; S^3] \cdot \#([S^9; S^3]) = 2^9 \cdot 3; \\
c_1 &= \#([P^3 \wedge P^3; S^3]) = 2^8 \cdot 3 \quad [4]; \\
c_2 &= \#([S^5 \wedge P^3; S^3]) = \#([S^9 \wedge P^2; S^3] \cdot \#([S^6; S^3]) = 4 \cdot 12 = 24 \cdot 3; \\
c_3 &= \# \pi_6 (S^3) = 2^2 \cdot 3.
\end{align*}
\]

Thus \( \#(X \wedge X; X) = 2^{86} \cdot 3^{14}. \)

A similar calculation would show that \( S^3 \times S^3 \) has \( 2^{20} \cdot 3^{14} \) homotopy classes of \( H \)-structures. This result was obtained earlier by Norman [7] using Loibel's Theorem [5].

**Bibliography**


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