ON THE GROUP INDICES OF THE PARASYMPLECTIC GROUP OF LEVEL F

TETSUO KODAMA

Abstract. We obtain the group indices of a parasymplectic group to a principal congruence subgroup of level q and to an inhomogeneous congruence subgroup of level q.

1. Introduction. Let \( \mathfrak{o} \) be the integer ring of an algebraic number field. Let \( \{ f_i \} \) be a set of \( n \) elements in \( \mathfrak{o} \) such that \( f_i \) is divisible by \( f_{i-1} \) for every \( i \) (\( 2 \leq i \leq n \)). Then we define \( f_{i,j} = f_i f_j \) for \( i \leq j \) and \( f_{i,j} = 1 \) for \( i > j \), and denote the set \( \{ f_{i,j} \} \) by \( F \). When all \( f_{i,j} = 1 \), then \( F \) is denoted by \( I \).

Let \( \mathfrak{o}_n^*(F) \) be the ring of matrices \( (f_{i,j} x_{i,j}) \) of degree \( n \), whose \( (i,j) \) element \( f_{i,j} x_{i,j} \) belongs to the ideal \( f_{i,j} \mathfrak{o} \) of \( \mathfrak{o} \) for every \( i,j \). Then we denote \( (f_{i,j} x_{i,j}) \), whose \( (i,j) \) element is \( f_{i,j} x_{i,j} \), by \( \hat{X} \) for any matrix \( X = (f_{i,j} x_{i,j}) \) in \( \mathfrak{o}_n^*(F) \).

Let \( \mathfrak{o}_n(F) \) be the ring of matrices \( (A \ B \ C \ D) \) of degree \( 2n \), whose submatrices \( A, B, C, D \) belong to \( \mathfrak{o}_n^*(F) \). We define a matrix \( \hat{M} \) in \( \mathfrak{o}_n(F) \) by

\[
\begin{pmatrix}
\hat{A} & \hat{C} \\
\hat{B} & \hat{D}
\end{pmatrix}
\]

associated with \( M = (A \ B \ C \ D) \) in \( \mathfrak{o}_n(F) \).

Now we shall define three groups in \( \mathfrak{o}_n(F) \) (see [1]); \( \Gamma(n,F) = \{ M | M \hat{M} J M = J, M \in \mathfrak{o}_n(F) \} \), where

\[
J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}
\]

and \( E_n \) is the unit matrix in \( \mathfrak{o}_n^*(F) \) (the parasymplectic group of level \( F \)), \( \Gamma_n = \{ M | M \in \Gamma(n,F) \) and \( M \equiv E_{2n} \text{ mod } q \} \) where \( q \) is an ideal in \( \mathfrak{o} \) (the principal congruence subgroup of level \( q \) of \( \Gamma(n,F) \)), \( \Gamma_q = \{ M | M \in \Gamma(n,F) \) and \( C \equiv 0 \text{ mod } q \} \), where \( M = (A \ B \ C \ D) \) (the inhomogeneous congruence subgroup of level \( q \) of \( \Gamma(n,F) \)). Then \( \Gamma(n,F,q) \) is a normal subgroup of \( \Gamma(n,F) \).

Received by the editors January 5, 1971.

AMS 1969 subject classifications. Primary 2065, 1545.

Key words and phrases. Parasymplectic group, principal congruence subgroup of level \( q \), inhomogeneous congruence subgroup of level \( q \).

1 This work was done while the author was at the University of Toronto, Toronto, Canada. The author acknowledges support received from the National Research Council of Canada, Grant No. A-7210.

© American Mathematical Society 1972
Our principal aims of the paper are to prove the following theorems:

**Theorem 1.** The group \( \Gamma(n, F)/\Gamma(n, F, q) \) is isomorphic to a group \( \Delta(n, F, q) \).

A definition of the group \( \Delta(n, F, q) \) is in §2 and a proof of the theorem is in §3 which is done by using a suitably modified argument of Klingen [3] and a “Hilfssatz” of Christian [1].

**Theorem 2.** For the group indices \( \mu(n, F, q) = \langle \Gamma(n, F): \Gamma(n, F, q) \rangle \) and \( \mu^0(n, F, q) = \langle \Gamma(n, F): \Gamma^0(n, F, q) \rangle \) it holds that

\[
\mu(n, F, q) = N(q)^{n(n+1)/2} \prod_{p \mid q} \prod_{i=1}^{n} (1 - N(p^{-2k_i+p})),
\]

\[
\mu^0(n, F, q) = N(q)^{n(n+1)/2} \prod_{p \mid q} \prod_{i=1}^{n} (1 + N(p^{-k_i+p})),
\]

where \( N(\cdot) \) is the norm of ideal, and \( k_{i,p} = k - i + 1 \) is the largest integer for a fixed integer \( i \) and a prime ideal \( p \) with \( \langle f_{ik}, p \rangle = 1 \).

From these theorems we shall get

**Corollary of Theorem 2.** The group \( \Delta(n, F, q) \) is isomorphic to the group \( \Delta(n, I, q) \), when \( \langle f_{i1}, q \rangle = 1 \) for every \( i \).

Proofs of Theorem 2 and the Corollary are in §4.

As our special cases we obtain \( \mu(n, I, q) \) and \( \mu^0(n, I, q) \) of Klingen [3], and those in which \( \mathfrak{o} \) is the rational integer ring (Koecher [4] or Satake [5]).

The author wishes to thank Professor G. de B. Robinson for his kind support of this work.

2. The group \( \Delta(n, F, q) \). Let us denote the residue ring \( f_{i1} \mathfrak{o}[f_{i2} \mathfrak{o}] \) by \( \mathfrak{o}_{ij} \) and let \( \mathfrak{o}^*_n(F, q) \) be the set of matrices of degree \( n \), \( (m_{ij}) \), with the element \( \bar{m}_{ij} \) in \( \mathfrak{o}_{ij} \).

We define an addition \( \oplus \) and a multiplication \( \otimes \) for any two elements \( (\bar{m}_{ij}) \) and \( (\bar{n}_{ij}) \) in \( \mathfrak{o}^*_n(F, q) \) by

\[
(\bar{m}_{ij}) \oplus (\bar{n}_{ij}) = (\{m_{ij} + n_{ij}\}), \quad (\bar{m}_{ij}) \otimes (\bar{n}_{ij}) = \left( \sum_{j=1}^{n} \bar{m}_{ij} \cdot \bar{n}_{jk} \right),
\]

where \( \bar{m}_{ij} \bar{n}_{jk} = m_{ij} n_{jk} \mod q \) for \( m_{ij} \in f_{ij} \mathfrak{o}, n_{jk} \in f_{jk} \mathfrak{o} \).

In particular it holds that, for \( \bar{m}_{i1}, \bar{m}_{i2} \in \mathfrak{o}_{ij} \) and \( \bar{n}_{i1}, \bar{n}_{i2} \in \mathfrak{o}_{ik} \),

\[
(\bar{m}_{i1} \bar{m}_{i2}) \cdot (\bar{n}_{i1} \bar{n}_{i2}) = (\bar{m}_{i1} \cdot \bar{n}_{i1}) (\bar{m}_{i2} \cdot \bar{n}_{i2}).
\]

It is obvious that these operations make sense and \( \mathfrak{o}^*_n(F, q) \) is a ring under the operations. Throughout the rest of this paper we shall denote simply \( \oplus \) by \( + \), and skip \( \otimes \) and \( \cdot \) to avoid complicated notations.
Now we define $\hat{X}$ attached to $X=((f_{ij}x_{ij})^{-})$ in $o^*(F, q)$ as in §1 by $\hat{X}=((f_{ij}x_{ij})^{-})$. This definition is independent of the choice of representatives $f_{ij}x_{ij}$ mod $f_iq_i$, and $\hat{X}$ is uniquely determined as an element of $o_n^*(F, q)$.

Let $o_n(F, q)$ be the set of matrices $M=(\begin{smallmatrix}A & B \\ C & D \end{smallmatrix})$ whose submatrices $A, B, C, D$ belong to $o_n^*(F, q)$, then the set forms a ring under the operations induced naturally from those in $o_n^*(F, q)$. For any $M$ in $o_n(F, q)$ we define $\hat{M}$ by

$$\begin{pmatrix} A & \hat{C} \\ B & \hat{D} \end{pmatrix}.$$

**Proposition 1.** Let us define $\Delta(n, F, q)=\{M|M\in o_n(F, q) \text{ and } \hat{M}JM=J\}$ for a fixed ideal $q$ of $o$; then $\Delta(n, F, q)$ is a group under the same multiplication as $o_n(F, q)$.

**Proof.** As $[MN]\hat{=}=\hat{N}\hat{M}$ for any $M, N$ in $o_n(F, q)$, $MN\in\Delta(n, F, q)$ if $M, N\in\Delta(n, F, q)$. The inverse element of $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ is

$$\begin{pmatrix} \hat{D} & -\hat{B} \\ -\hat{C} & \hat{A} \end{pmatrix}.$$

**Remark.** $(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$ is in $\Gamma(n, F)$ (or $\Delta(n, F, q)$) if and only if $\hat{A}\hat{D}=\hat{C}\hat{B}=E$ and $\hat{A}\hat{C}=\hat{C}\hat{A}$, $\hat{B}\hat{D}=\hat{D}\hat{B}$ in $o_n^*(F)$ (or in $o_n^*(F, q)$).

3. **Proof of Theorem 1.** Let $\phi_{ij}$ be a natural homomorphism of $f_{ij}$ onto $o_{ij}$ and $\phi^*$ be a mapping of $o_n^*(F)$ such that for any $X=(f_{ij}x_{ij})\in o_n^*(F)$, $\phi^*(X)=(\phi_{ij}(f_{ij}x_{ij}))$. Let us define a mapping $\phi$ of $o_n(F)$ by

$$\phi(M) = \begin{pmatrix} \phi^*(A) & \phi^*(B) \\ \phi^*(C) & \phi^*(D) \end{pmatrix},$$

where $M=(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix})$, and $A=(f_{ij}a_{ij}), \cdots$. We can easily check by considering §2 that $\phi$ is a homomorphism of $o_n(F)$ onto $o_n(F, q)$, and $\phi$ induces a homomorphism of $\Gamma(n, F)$ into $\Delta(n, F, q)$ whose kernel is $\Gamma(n, F, q)$.

In the following we shall prove the ontoness of $\phi$ of $\Gamma(n, F)$ by using induction on $n$.

When $n=1$, then $F=I$, which leads to a well-known result (Klingen [3], or Hurwitz [2]). So we assume that our proposition is true for any positive integer less than $n$. Let

$$\hat{M} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}.$$

* Except when confusion might arise, we use the same symbols for $1, E, J$, etc. both in $o$ and $o_{ij}$, $(i\geq j)$ or both in $o_n(F)$ ($o_n^*(F)$) and $o_n(F, q)$ ($o_n^*(F, q)$).
be in $\Delta(n, F, q)$, and let $M$ be in $\mathcal{O}_n(F)$ and be one of the inverse images of $\overline{M}$. By the remark of Proposition 1, $[\overline{A}]^\perp \overline{B} = \overline{E}$ holds and so the $(1, 1)$ element of the left-hand side is 1,

$$\sum_{j=1}^n \{ \phi_{11}(f_{1j}a_{1j}d_{1j}) - \phi_{11}(f_{1j}c_{1j}b_{1j}) \} = 1.$$ 

Therefore

$$\langle a_{11}, f_{12}a_{21}, \ldots, f_{1n}a_{n1}, c_{11}, \ldots, f_{1n}c_{n1}, q \rangle = 1,$$

where $\langle \cdot \cdot \cdot \rangle$ means the largest common divisor. If $a_{11} = \cdots = a_{n1} = c_{11} = \cdots = c_{n1} = 0$, we get

$$T_SM = \begin{pmatrix} a_{11} \\ \vdots \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} \quad \text{for} \quad T_S = \begin{pmatrix} 1 & E_{n-1} \\ E_{n-1} \end{pmatrix} \in \Gamma(n, F),$$

so we may assume that the set $(a_{21}, \cdots, a_{n1}, c_{11}, \cdots, c_{n1})$ contains at least a nonzero element. Then by Hilfssatz 1 of Klingon [3], there exists $x$ in $q$, with $\langle a_{11} + x, f_{12}a_{21}, \cdots, f_{1n}a_{n1}, c_{11}, f_{12}c_{21}, \cdots, f_{1n}c_{n1} \rangle = 1$, so that we can find $x_1, \cdots, x_n, y_1, \cdots, y_n$ in $\mathcal{O}$ satisfying

$$x_1(a_{11} + x) + \sum_{i=2}^n x_if_{1i}a_{1i} + \sum_{i=1}^n y_if_{1i}c_{1i} = 1.$$ 

Since $\langle x_1, f_{12}x_2, \cdots, f_{1n}x_n, y_1, \cdots, f_{1n}y_n \rangle = 1$, therefore by Hilfssatz 3 of Christian [1] there exists a matrix $N_1$ in $\Gamma(n, F)$ whose first row is $(x_1, f_{12}x_2, \cdots, f_{1n}x_n, y_1, \cdots, f_{1n}y_n)$. For this $N_1, N_1M \equiv (1, \ast, \ast) \mod (q, F)$. According for a suitable matrix $N_2 \in \Gamma(n, F)$,

$$N_2M \equiv \begin{pmatrix} 1 & * & * & * \\ A_1 & * & B_1 \\ C_1 & * & D_1 \end{pmatrix} \mod (q, F).$$

\* A very small modification is necessary because $\mathcal{O}$ is the rational integer ring in this Hilfssatz. We can do it by using Hilfssatz 1, 2, 3 of Klingon [3].

\* We use this notation in the following sense: "$L \equiv N \mod (q, F)$ for $L, N \in \mathcal{O}(F)$" is equivalent to "$\phi(L) = \phi(N)$".
Let $F_1 = \{f_{ij} \mid f_{ij} \in F, 2 \leq i, j \leq n\}$ and $\phi_1$ be a homomorphism induced by $\phi$ of $\mathfrak{o}_{n-1}(F_1)$ onto $\mathfrak{o}_{n-1}(F_1, q)$. Then

$$\phi_1 \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

is in $\Delta(n-1, F_1, q)$, since $\phi(N_2 M) = \phi(N_2)\phi(M)$ is in $\mathfrak{o}_n(F, q)$.

By our inductive assumption there exists a matrix $N_3^*$ in $\Gamma(n-1, F_1)$ such that

$$\phi_1(N_3^*) = \phi_1 \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$ 

Thus we can find $N_3$ in $\Gamma(n, F)$ such that

$$N_3 N_2 M \equiv \begin{pmatrix} 1 & a \\ & S \\ & E_{n-1} \\ & 1 \\ b & E_{n-1} \end{pmatrix} \mod (q, F),$$

where $a = (f_{12} a_{12}', \ldots, f_{1n} a_{1n}'), b = (d_{21}', \ldots, d_{n1}')$ and

$$S = \begin{pmatrix} b'_{11} & \cdots & f_{1n} b'_{1n} \\ \vdots & \ddots & \vdots \\ \vdots & & 0 \\ b'_{n1} \end{pmatrix}$$

with $a'_{1i} + d'_{1i} \equiv 0 \mod q$ and $b'_{1i} \equiv b'_{1i} \mod q$.

We conclude that there exists a matrix $N_4$ in $\Gamma(n, F)$ such that

$$N_4 N_3 N_2 M \equiv \begin{pmatrix} E & S^* \\ & E \end{pmatrix} = T_{S^*} \mod (q, F),$$

where $T_{S^*}$ is in $\Gamma(n, F)$ with $S^* = S^*$. Set $N_6 = N_2^{-1} N_3^{-1} N_4^{-1} T_{S^*}$, then $N_6 \in \Gamma(n, F)$ and $\bar{M} = \phi(M) = \phi(N_6)$. This shows that $\phi$ is a homomorphism of $\Gamma(n, F)$ onto $\Delta(n, F, q)$, which completes our proof.

4. Proofs of Theorem 2 and its Corollary. To simplify our problem we prove

**Proposition 2.** If $q = q_1 q_2$ with $\langle q_1, q_2 \rangle = 1$, then $\Delta(n, F, q) = \Delta(n, F, q_1) \otimes \Delta(n, F, q_2)$, where $\otimes$ means the direct product of the groups.
PROOF. For any \( \phi(M) \in \Delta(n, F, q) \) with \( M \in \Gamma(n, F) \), let us associate a pair of matrices \( M_k \in \mathfrak{a}_q(F) \) such that \( M_k \equiv M \mod(q_k, F), k=1, 2 \). This induces an isomorphism of \( \Delta(n, F, q) \) into \( \Delta(n, F, q_k) \otimes \Delta(n, F, q_2) \). Conversely for any pair \( \phi_k(M_k) \in \Delta(n, F, q_k) \) with \( M_k \in \Gamma(n, F), k=1, 2 \), we determine \( M \in \mathfrak{a}_q(F) \) such that \( M_k \equiv M \mod(q_k, F), k=1, 2 \). \( M = q_1 M_2 + q_2 M_1 \) is a solution of the congruence equation, where \( q_1 + q_2 = 1 \) with \( q_k \in q_k \). For this \( M \) we have \( \phi(MJM) = J \). This shows that the isomorphism is onto and completes our proof.

COROLLARY. If \( q = \prod_{p \mid q} p^e \), \( p \) is a prime ideal, then

\[
\Delta(n, F, q) = \prod_{p \mid q} \otimes \Delta(n, F, p^e).
\]

By Theorem 1 and the Corollary of Proposition 2, \( \mu(n, F, q) = \) the order of \( \Delta(n, F, q) = \prod_{p \mid q} \mu(n, F, p^e) \). Therefore we shall consider only \( \mu(n, F, p^e) \) and \( \Delta(n, F, p^e) \).

Let us assume that \( \langle f_{1k}, p \rangle = 1 \) but \( \langle f_{1k+1}, p \rangle \neq 1 \). If the transposed vector of the first column of \( M \in \Delta(n, F, p^e) \) is \( (\bar{a}_{11}, \ldots, \bar{c}_{n1}, \bar{a}_{11}, \ldots, \bar{c}_{n1}) \), then as already stated

\[
\langle \bar{a}_{11}, \ldots, f_{1k} \bar{a}_{k1}, \bar{c}_{11}, \ldots, f_{1k} \bar{c}_{k1}, p/p^e \rangle = 1
\]

with \( [f_{1k}a_{11}]^{-1} \in \mathfrak{o}_{11} \), and by the above assumption this is equivalent to

\[
\langle \bar{a}_{11}, \ldots, f_{1k} \bar{a}_{k1}, \bar{c}_{11}, \ldots, f_{1k} \bar{c}_{k1}, p/p^e \rangle = 1.
\]

Assume (i) \( \langle \bar{a}_{11}, p/p^e \rangle \neq 1 \) for \( 1 \leq i \leq m-1 \), but \( \langle \bar{a}_{m1}, p/p^e \rangle = 1 \) for \( m \leq k \), or (ii) \( \langle \bar{a}_{11}, p/p^e \rangle \neq 1 \) for \( 1 \leq i \leq k \), \( \langle \bar{a}_{m1}, p/p^e \rangle = 1 \) for \( k < i \leq m-1 \), but \( \langle \bar{a}_{m1}, p/p^e \rangle = 1 \) for \( m \leq k \).

Case (i). We determine \( [f_{1m} x_m]^{-1} \in \mathfrak{o}_{1m} \) with \( [f_{1m} x_m]^{-1} = 1 \), and put

\[
U_i = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{11} \end{pmatrix} ; \quad V_{11} = \begin{pmatrix} 0 & [f_{1m} x_m]^{-1} \\ \bar{a}_{m1} & 0 \end{pmatrix} , \quad V_{11} V_{12} = E_n.
\]

\( U_i \in \Delta(n, F, p^e) \) and \( U_1 M = (\begin{smallmatrix} A & 1 \\ \end{smallmatrix} \begin{smallmatrix} B & * \\ \end{smallmatrix}) \). With a suitable \( L \in \Delta(n, F, p^e) \),

\[
LU_1 M = \begin{pmatrix} 1 & * & b_1 \\ A_1 & b_2 & B_1 \\ 1 \\ C_1 & b_1 & D_1 \end{pmatrix} ; \quad \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \in \Delta(n-1, F_1, p^e).
\]
Moreover with $U_2$ and $T_S$ of $\Delta(n, F, p^e)$, we have

$$LU_1MU_2T_S = \begin{pmatrix} 1 & A_1 & B_1 \\ & 1 \\ & C_1 & D_1 \end{pmatrix},$$

where $U_2 = \begin{pmatrix} V_{21} \\ V_{22} \end{pmatrix}$,

with

$$V_{21} = \begin{pmatrix} 1 \\ -a \\ E_{n-1} \end{pmatrix}$$

and $V_{21}V_{22} = E_n$, $T_S = \begin{pmatrix} E & S \\ E \end{pmatrix}$ with

$$S = \begin{pmatrix} * & -b_1 \\ -b_1 & 0 \end{pmatrix}.$$

**Case (ii).** We determine $[f_{1m}x_m]^{-e_0}_{-m}$ with $[f_{1m}x_m]^{-e_{m1}}_{-m1} = 1$, and put

$$U_1 = \begin{pmatrix} 0 & E_m \\ -[f_{1m}x_m]^{-} & 0 \\ [f_{1m}c_{m1}]^{-} & -x_m \\ \hat{c}_{m1} & 0 & E_m \\ \hat{c}_{m1} & 0 & E_{n-m-2} \end{pmatrix}.$$

Then $U_1$ is in $\Delta(n, F, p^e)$ and $U_1M = \begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$. So that, by the same method as in (i), we obtain the same result. In these transformations $L$ and $U_1$ are unique, and $a, b_1$ are uniquely determined by given matrices

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix},$$

$\mathbf{b}$ and $\mathbf{b}_2$.

So we have a recurrence formula

$$\mu(n, F, p^e) = N(p^{e(4n-1)})(1 - N(p^{-2k}))\mu(n - 1, F_1, p^e),$$

where $N(\cdot \cdot \cdot)$ is the norm of the ideal. Considering the well-known relation (A. Hurwitz [2], or H. Klingen [3]), $\mu(1, 1, p^e) = N(p^{e^e})(1 - N(p^{-2}))$, we obtain

$$\mu(n, F, p^e) = N(p^{e(n(2n+1))}) \prod_{i=1}^{n} (1 - N(p^{-2k_i}))),$$

where $k_i = k_i' - i + 1$ is the largest integer with $\langle f_{ik_i}, p \rangle = 1$ for fixed $i$ ($1 \leq i \leq n$).
Therefore for a general ideal \( q \),

\[
\mu(n, F, q) = N(q)^{n(2n+1)} \prod_{p|q} \prod_{i=1}^{n} (1 - N(p^{-2k_{i,p}})),
\]

where \( k_{i,p} \) means the above explained \( k_i \) for \( p \).

The same method as previously used can be applied to obtain the order \( \nu(n, F, q) \) of \( \Gamma^q(n, F, q)/\Gamma(n, F, q) \). By Theorem 1, \( \Gamma^q(n, F, p^e)/\Gamma(n, F, p^e) \) is isomorphic to a subgroup of \( \Delta(n, F, p^e) \) consisting of matrices of such a form \( (\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}) \). So with suitable \( L, R \in \Delta(n, F, p^e) \) we obtain

\[
LMR = \begin{pmatrix}
1 & A_1 & B_1 \\
& 1 & B_1 \\
& & D_1
\end{pmatrix}
\]

with \( \begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix} \in \Delta(n - 1, F_1, p^e) \),

and so

\[
\nu(n, F, p^e) = N(p^{e(n-1)})(1 - N(p^{-k}))\nu(n - 1, F_1, p^e).
\]

Considering \( \nu(1, 1, p^e) = N(p^{2e})(1 - N(p^{-1})) \), we have

\[
\nu(n, F, p^e) = N(p^{e(n+1)/2}) \prod_{i=1}^{n} (1 - N(p^{-k_i})).
\]

So for a general ideal \( q \),

\[
\nu(n, F, q) = N(q^{n(3n+1)/2}) \prod_{p|q} \prod_{i=1}^{n} (1 - N(p^{-k_{i,p}})).
\]

Therefore

\[
\mu(q(n, F, q) = \mu(n, F, q) \nu(n, F, q) = N(q^{n(n+1)/2}) \prod_{p|q} \prod_{i=1}^{n} (1 + N(p^{-k_{i,p}})),
\]

and we have proved Theorem 2.

**Proof of the Corollary of Theorem 2.** From the definition of \( k_{i,p} \), it follows that \( k_{i,p} = n - i + 1 \), accordingly,

\[
\mu(n, I, q) = \mu(n, F, q) = N(q)^{n(2n+1)} \prod_{p|q} \prod_{i=1}^{n} (1 - N(p^{-2i})).
\]

Let

\[
T = \begin{pmatrix}
1 & f_{12} & \cdots \\
& \ddots & \ddots \\
& & \ddots & f_{1n}
\end{pmatrix}, \quad T \in \mathfrak{a}_n^*(F),
\]
then \( \phi^*(T) \in \mathfrak{o}_n^*(F, q) \) and \( \phi_{ii}(f_{ii}) \) is a unit in \( \mathfrak{o}_{ii} \) for all \( i \). Therefore \( \phi(T, E) \Delta(n, F, q)(\phi(T, E))^{-1} \) is an isomorphic image in \( \Delta(n, I, q) \) of \( \Delta(n, F, q) \). Comparing the orders of both groups, it follows that \( \Delta(n, F, q) \) and \( \Delta(n, I, q) \) are isomorphic.

**References**


**Department of Mathematics, College of General Education, Kyushu University, Fukuoka, Japan**