INJECTIVE HULLS OF CERTAIN S-SYSTEMS
OVER A SEMILATTICE

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Abstract. We construct, in the category of S-systems over a semilattice, the injective hulls of S-systems which are homomorphic images of S-subsystems of S.

1. Introduction. In [1] Berthiaume showed that injective hulls exist in the category of S-systems (or S-sets) over a semigroup S. In that paper he also showed that if S is a chain then the injective hull of S itself is its Dedekind-MacNeile completion. In the present paper we consider the case where S is a semilattice and construct the injective hulls of S-systems which are homomorphic images of S-subsystems of S (or, in the notation of [3], S-systems which are in HS(S)). We do this by adapting the techniques used by Bruns and Lakser in [2] to construct injective hulls in the category of semilattices. We obtain as corollaries Berthiaume's result for chains, a characterization of injective cyclic S-systems over a semilattice, and the result that a semilattice S is injective in the category of semilattices if and only if it is injective in the category of S-systems.

2. Preliminaries. Let S be a semigroup. A (right) S-system is a set M equipped with a map (written multiplicatively) from M x S to M such that m(s_1 s_2) = (m s_1) s_2 for all m in M and all s_1, s_2 in S. If one thinks of each element of S as inducing a unary operation on an S-system M, then M is a finitary algebra and all the notions of universal algebra are available. Thus if M and N are S-systems we have A is an S-subsystem of M if and only if A is a homomorphism if and only if \phi(m s) = \phi(m) s for all m in M and all s in S, and an equivalence relation \sim on M is a congruence relation if and only if m_1 \sim m_2 implies m_1 s \sim m_2 s for all s in S. Unless otherwise stated, all algebraic notions will be in this category. We will assume throughout that the semigroup S is a semilattice (i.e., commutative and idempotent).

Lemma 1. If an S-system M has the property that MS = M, then it is partially ordered by the rule m_1 \preceq m_2 if and only if m_1 = m_2 s for some s in S.

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Proof. For each \( m \in M \) we have \( m = m_s \) for some \( m \in M \) and \( s \in S \) (since \( MS = M \)), and hence \( m = ms \), so \( m \leq m \). If \( m_1 \leq m_2 \) and \( m_2 \leq m_1 \), we have \( s_1, s_2 \in S \) such that \( m_1 = m_{s_1} \) and \( m_2 = m_{s_2} \). Now \( m_1 s_1 = (m_1 s_2) s_1 = m_2 s_1 s_2 s_1 = m_2 s_2 s_1 = m_1 s_2 = m_2 \). Transitivity is obvious.

Notice that if \( S \) has an identity and \( M \) is a unitary \( S \)-system, then \( MS = M \) and Lemma 1 applies.

When we are dealing with a partial order on an \( S \)-system we will use the symbols "\( V \)" and "\( \wedge \)" to denote least upper bounds and greatest lower bounds, respectively.

We will refer to the partial order of Lemma 1 as the natural partial order on \( M \).

If an \( S \)-system \( M \) is partially ordered in some way and if \( A \subseteq M \) is such that \( V A \) exists, we will say that \( V A \) is \( S \)-distributive if and only if, for each \( S \in S \), \( V \{ sa \mid a \in A \} \) exists and equals \( (V A) s \).

Recall the following definitions in a category of algebras: An algebra \( C \) is injective if and only if every homomorphism from a subalgebra \( A \) of an algebra \( B \) into \( C \) has an extension to all of \( B \). An extension \( C \) of an algebra \( A \) is essential if and only if any homomorphism from \( C \) to an algebra \( B \), whose restriction to \( A \) is one-to-one, is itself one-to-one. An injective hull of an algebra is an essential, injective extension.

Lemma 2. Let \( C \) be an \( S \)-system which is partially ordered in such a way that \( V A \) exists, \( V A \) is \( S \)-distributive if and only if, for each \( S \in S \), \( V \{ sa \mid a \in A \} \) exists and equals \( (V A) s \).

We will call a subset \( A \) of a poset \( C \) join-dense in \( C \) if and only if \( c = V \{ a \mid a \leq c \} \) for each \( c \in C \). If \( C \) is a complete lattice in which arbitrary joins are \( S \)-distributive, then \( C \) is injective.

Proof. Let \( A \) be an \( S \)-subsystem of an \( S \)-system \( B \) and let \( \phi : A \rightarrow C \) be a homomorphism. Define \( \phi^* : B \rightarrow C \) by

\[
\phi^*(b) = V \{ \phi(a) \mid a \in A, a = bs \text{ for some } s \in S \}.
\]

If \( b \in A \), then

\[
\phi^*(b) = V \{ \phi(bs) \mid s \in S \} = V \{ \phi(b)s \mid s \in S \} = \phi(b)
\]

and thus \( \phi^* \) extends \( \phi \). If \( s_0 \in S \) it is easy to see that \( \{ as_0 \mid a \in A, a = bs \text{ for some } s \in S \} = \{ a \mid a \in A, a = bs_0s \text{ for some } s \in S \} \). Thus

\[
\phi^*(bs_0) = (V \{ \phi(a) \mid a \in A, a = bs \text{ for some } s \in S \})s_0
\]

\[
= V \{ \phi(a)s_0 \mid a \in A, a = bs \text{ for some } s \in S \}
\]

\[
= V \{ \phi(as_0) \mid a \in A, a = bs \text{ for some } s \in S \}
\]

\[
= V \{ \phi(a) \mid a \in A, a = bs_0s \text{ for some } s \in S \} = \phi^*(bs_0).
\]

We will call a subset \( A \) of a poset \( C \) join-dense in \( C \) if and only if \( c = V \{ a \mid a \leq c \} \) for each \( c \in C \). If \( A \) and \( C \) are also \( S \)-systems we will say that \( S \)-distributive joins in \( A \) are preserved in \( C \) if and only if \( a = V B \)
whenever \( B \subseteq A \) and \( a = \bigvee_A B \) is \( S \)-distributive. We will call a map \( \phi \) on a poset \( P \) decreasing if and only if \( \phi(a) \leq a \) for all \( a \in P \).

**Lemma 3.** Let \( C \) be an \( S \)-system which is partially ordered in such a way that the unary operations induced by \( S \) preserve the order and are decreasing. Let \( A \) be an \( S \)-subsystem of \( C \) and suppose that for each \( a \in A \) there is an \( s_a \in S \) such that, for each \( c \in C \), \( c \land a \) exists and equals \( cs_a \). If \( A \) is join-dense in \( C \) and if \( S \)-distributive joins in \( A \) are preserved in \( C \), then \( C \) is an essential extension of \( A \).

**Proof.** Let \( \phi : B \rightarrow C \) be a homomorphism with \( \phi \mid_A \) one-to-one. If \( \phi \) is not one-to-one there exist elements \( a, b \in C \) with \( a \neq b \) and \( \phi(a) = \phi(b) \). Since \( A \) is join-dense in \( C \) we may suppose there exists \( u \in A \) with \( u \leq b \) and \( u \leq a \). We have \( \phi(a \land u) = \phi(as_u) = \phi(a)s_u = \phi(b)s_u = \phi(bs_u) = \phi(b \land u) = \phi(u) \).

Now suppose \( s \in S \) and let \( M = \{(u \land x) \mid x \leq a, x \in A \} \). If we show that \( u \subseteq \bigvee_A M \) we will have shown (considering the special case \( s = s_a \)) that \( u \subseteq \bigvee \{u \land x \mid x \leq a, x \in A \} \) and is an \( S \)-distributive join. Hence \( u = \bigvee_C \{u \land x \mid x \leq a, x \in A \} \leq a \), a contradiction. Since \( u \land x \leq u \) implies \( (u \land x)s \leq us \), it is clear that \( u \subseteq u \) is an upper bound for \( M \). Let \( v \in A \) be another upper bound for \( M \) with \( v \neq u \). Since meets exist in \( A \) we may further assume that \( v \prec u \). If \( c \in A \) and \( c \leq (u \land a) \) we have \( c \subseteq u \) and \( c = us \land c = us \) implies \( c \subseteq u \land c = (u \land a)s = (u \land c)s \) with \( c \leq as \leq a \). Hence we can again use the fact that \( A \) is join-dense in \( C \) and obtain

\[
(u \land a)s = \bigvee_C \{(u \land x)s \mid x \leq a, x \in A \} = \bigvee_C M \leq v.
\]

Now we have

\[
\phi(us) = \phi(u)s = \phi(a \land u)s = \phi((a \land u)s) = \phi((a \land u)s \land v) = \phi((a \land u)ss_u) = \phi(a \land u)ss_u = \phi(u)ss_u = \phi(uss_u) = \phi(us \land v) = \phi(v),
\]

a contradiction. This establishes the fact that \( us \subseteq \bigvee_A M \) and finishes the proof.

3. **Injective hulls.** Let \( M \) be an \( S \)-system such that \( MS = M \). Recall that, by Lemma 1, \( M \) is partially ordered by the rule \( m_1 \leq m_2 \) if and only if \( m_1 \leq m_2 s \) for some \( s \in S \). Following Bruns and Lakser we will call a subset \( N \) of \( M \) admissible if and only if \( \bigvee N \) exists and is \( S \)-distributive, and we will call \( N \) a \( D \)-ideal if and only if \( \forall y \in N \) and \( x \leq y \) imply \( x \in N \) (i.e., \( NS \subseteq N \)) and \( N \) is closed under \( S \)-distributive joins (i.e., \( A \subseteq N \) and \( A \) admissible implies \( \bigvee A \subseteq N \)). Now \( I_D(M) \), the set of all \( D \)-ideals of \( M \), is closed under arbitrary intersections and is thus a complete lattice under set inclusion. An obvious modification of the proof of [2, Lemma 3] shows that the join operation in \( I_D(M) \) is given by

\[
\bigvee \{A_i \mid i \in I\} = \{\bigvee N \mid N \subseteq \bigcup \{A_i \mid i \in I\}, N \text{ admissible}\}.
\]
It is easy to show that if $N$ is a $D$-ideal of $M$ then $N^S = \{ ns | s \in S \}$ is also a $D$-ideal and that $N^S = N \cap M^S$. Thus $I_D(M)$ is a complete lattice in which arbitrary joins are $S$-distributive. Notice that $m^S = \{ x \in M | x \leq m \}$, that these principal ideals are clearly $D$-ideals and that $m \rightarrow m^S$ is an embedding of $M$ in $I_D(M)$. Now, considering $M$ as an $S$-subsystem of $I_D(M)$, notice that $S$-distributive joins in $M$ are preserved in $I_D(M)$.

It is clear that $S$ itself is an $S$-system and we now restrict our attention to $HS(S)$, that is, to $S$-systems which are of the form $A/\sim$ where $A$ is an $S$-subsystem of $S$ and $\sim$ is a congruence relation on $A$. Notice that $A$ is an ideal of $S$ and $\sim$ is a semigroup congruence on $A$ (since we have assumed $S$ to be commutative) and thus $A/\sim$ is a semilattice as well as an $S$-system. It is easy to see that $(A/\sim)S = A/\sim$ and that the partial order on $A/\sim$ as a semilattice coincides with the natural partial order of Lemma 1.

**THEOREM.** If $M \in HS(S)$, then $I_D(M)$ is the injective hull of $M$.

**Proof.** $M = A/\sim$ where $A \subseteq S$ is an ideal and $\sim$ is a congruence relation on $A$. Denoting arbitrary elements of $A/\sim$ by $[x]$ with $x \in A$, we have that $[a]S = Ma$ since $[a] = [as] = [asa] = [as]a$ and $[x]a = [xa] = [ax] = [a]x$. Since a $D$-ideal $N$ is the join of the principal ideals it contains we have

$$N = \bigvee \{ [a]S \ | \ [a] \in N \} = \bigvee \{ N \cap Ma \ | \ [a] \in N \} \leq \bigvee \{ N \cap Ms \ | \ s \in S \} = \bigvee \{ Ns \ | \ s \in S \} \subseteq N.$$ 

Thus $N = \bigvee \{ Ns | s \in S \}$ for each $N \in I_D(M)$ so the hypotheses of Lemma 2 are satisfied and $I_D(M)$ is injective. Since the unary operations in $I_D(M)$ are given by $N^S = N \cap M^S$, for each $s \in S$, it is apparent that they preserve the order and are decreasing and that for each $[a] \in M$ we have $Na = N \cap Ma = N \cap [a]S$. Thus, by identifying $M$ with the $S$-subsystem of $I_D(M)$ consisting of the principal order ideals of $M$, we see that the hypotheses of Lemma 3 are satisfied and that $I_D(M)$ is an essential extension of $M$.

**Corollary 1.** If $M \in HS(S)$, then $M$ is injective if and only if it is a complete lattice in which arbitrary joins are $S$-distributive.

**Proof.** $M$ is injective if and only if the embedding $m \rightarrow m^S$ of $M$ in $I_D(M)$ is onto. This is true precisely when every $D$-ideal of $M$ is principal. Clearly this is the case when $M$ is a complete lattice in which arbitrary joins are $S$-distributive. Conversely, if every $D$-ideal is principal, then the partial ordering of $I_D(M)$ by set inclusion (under which $I_D(M)$ is a complete lattice with $S$-distributive joins) coincides with its natural partial order as an $S$-system, i.e., $m_1S \leq m_2S$ if and only if $m_1 = m_2s$ for some $s \in S$. Since in this case $M$ is isomorphic to $I_D(M)$, $M$ is also a complete lattice with $S$-distributive joins.
Corollary 2 (Berthiaume). If $S$ is a chain, then its injective hull is its Dedekind-MacNeile completion.

Proof. If $S$ is a chain, then every order ideal is a $D$-ideal and hence $ID(S)$ is the Dedekind-MacNeile completion.

Corollary 3. A semilattice $S$ is injective in the category of semilattices if and only if it is injective in the category of $S$-systems.

Proof. By Corollary 1, $S$ is injective in the category of $S$-systems if and only if it is a complete lattice with the property that $(\bigvee M)\wedge s = \bigvee \{m\wedge s \mid m \in M\}$ for all $s \in S$, $M \subseteq S$. By [2, Theorem 1] these properties characterize injectivity in the category of semilattices.

Corollary 4. A cyclic $S$-system is injective if and only if it is a complete lattice (in its natural partial order) in which arbitrary joins are $S$-distributive.

Proof. If $M$ is a cyclic $S$-system, then $M = xS$ for some $x \in M$. It is clear that $MS = M$, so $M$ has a natural partial order (Lemma 1). Define a congruence relation on $S$ by $s_1 \sim s_2$ if and only if $xs_1 = xs_2$. The map $xS \rightarrow [s]$ is an isomorphism between $M$ and $S/\sim$ and hence $M \in HS(S)$ and Corollary 1 applies.

Corollary 5. Let $M$ be an $S$-system such that $MS = M$. If for each $m \in M$ there exists an $s \in S$ such that $mS = Ms$, then $M$ is injective if and only if it is a complete lattice in which arbitrary joins are $S$-distributive.

Proof. Define a congruence relation on $S$ by $s_1 \sim s_2$ if and only if $Ms_1 = Ms_2$. The map $m \rightarrow [s]$, where $mS = Ms$, is an isomorphism between $M$ and an $S$-subsystem of $S/\sim$. Since $SH(S) \subseteq HS(S)$ by [3, Theorem 1, p. 152], $M \in HS(S)$ and Corollary 1 applies.

References


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