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THE CLOSED IDEALS OF SOME DIRICHLET AND HYPO-DIRICHLET ALGEBRAS

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ABSTRACT. We characterize the closed ideals of (i) the Dirichlet algebras discovered by A. Browder and J. Wermer and (ii) the hypo-Dirichlet algebras discovered recently by A. G. Brandstein. Our results show that the class of closed ideals of each of these algebras is surprisingly restricted.

We consider algebras of functions defined by means of singular maps. A homeomorphism ψ of one circle onto another of radius r is called *singular* if ψ maps a Borel set of Lebesgue measure zero onto a set of measure $2\pi r$. If K is a compact subset of C with boundary ∂K, we denote by A_K the subalgebra of $C(\partial K)$ of all functions which admit continuous extensions to K, analytic on the interior of K.

Let $\Delta=\{z\in C:|z|\leqq 1\}$, the closed unit disc, $T=\partial\Delta$, and $A_0=A_\Delta$. Let q be a singular homeomorphism of T onto itself. Define $A_0(q)=\{f\in A_0: f\circ q\in A_0\}$. Then A is a Dirichlet algebra (i.e. $\mathrm{Cl}[\mathrm{Re}\, A_0(q)]=C_R(T)$) [3], and a maximal closed subalgebra of A_0 [1].

THEOREM 1. *I is a closed ideal of $A_0(q)$ if and only if there exist closed ideals I_1 and I_2 of A_0 such that $I=\{f\in I_1: f\circ q\in I_2\}$.*

Suppose now that $q\circ q=$identity. Let T/q denote the quotient space of T induced by q. Set $A_q=\{f\in A_0: f\circ q=f\}$. The subalgebra of $C(T/q)$ induced by A_q is a Dirichlet algebra on T/q, as well as a maximal closed subalgebra of $C(T/q)$ [3].

THEOREM 2. *I is a closed ideal of A_q if and only if there exists a closed ideal J of A_0 such that $I=A_q\cap J$.*

Let Γ be the annulus $\{z\in C:1\leqq|z|\leqq 2\}$, let $T'=\{z\in C:|z|=2\}$, and let $B_0=A_\Gamma$. Let p be a singular homeomorphism of T onto T' which is orientation preserving. Define $B_p=\{f\in B_0: f(z)=f(p(z)),\text{ all } z \text{ in } T\}$. The restriction of B_p to T is a hypo-Dirichlet algebra (i.e. there exist $f_1,\cdots,f_n$ in B_p whose analytic extensions to Γ are never zero such that the real vector space spanned by $\mathrm{Re}\, B_p$ and $\log|f_1|,\cdots,\log|f_n|$ is dense in $C_R(T)$) [2].

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THEOREM 3. *I is a closed ideal of B_p if and only if there exists a closed ideal J of B_0 such that $I=B_p\cap J$.*

We identify the dual space $C(X)^*$ of $C(X)$, X compact, with the space of all complex, regular, Borel measures on X. If A is a subspace of $C(X)$, $A^\perp$ is the space of all $\mu\in C(X)^*$ such that $\int f\,d\mu=0$ for each $f\in A$.

If A is a closed subalgebra of $C(X)$ and I is a closed ideal of a subalgebra of A, we let $I[A]$ denote the closed ideal of A generated by the functions in I.

LEMMA 1. *Let A and B be closed subalgebras containing 1 of $C(X)$. Suppose there exists a Borel subset E of X such that $|\nu|(E)=|\mu|(X-E)=0$ for every $\mu\in A^\perp$, $\nu\in B^\perp$. If I_1 and I_2 are closed ideals of A and B respectively, then $I_1\cap I_2$ is a closed ideal of $A\cap B$, and all the closed ideals of $A\cap B$ arise in this way.*

PROOF. Let I be a closed ideal of $A\cap B$. We shall show that $I=I[A]\cap I[B]$. Clearly $I\subseteq I[A]\cap I[B]$. To prove the reverse containment we shall prove $I^\perp\subseteq(I[A]\cap I[B])^\perp$. Thus, let $\eta\in I^\perp$ and $f\in I[A]\cap I[B]$. We have to show $\int f\,d\eta=0$.

Since $I[A]=\mathrm{Cl}(A\cdot I)$ and $I[B]=\mathrm{Cl}(B\cdot I)$, there exist sequences $\{g_n\}$ and $\{h_n\}$, $g_n\in A\cdot I$, $h_n\in B\cdot I$, which converge uniformly to f. Since

$$\int f\,d\eta=\lim_{n\to\infty}\int_E g_n\,d\eta+\lim_{n\to\infty}\int_{X-E}h_n\,d\eta$$

it suffices to show $\int_E g_n\,d\eta$ and $\int_{X-E}h_n\,d\eta$ are zero for each n.

Since $g_n\in A\cdot I$, $g_n=\sum_{r=1}^p a_ri_r$, where $a_r\in A$, and $i_r\in I$. Observe that $i_r\eta\in(A\cap B)^\perp$. It is known that $(A\cap B)^\perp=A^\perp+B^\perp$ [3, p. 547]; hence $i_r\eta=\mu_r+\nu_r$, $\mu_r\in A^\perp$, $\nu_r\in B^\perp$. Then

$$\int_E g_n\,d\eta=\sum_{r=1}^p\left(\int_E a_r\,d\mu_r+\int_E a_r\,d\nu_r\right)=\sum_{r=1}^p\int a_r\,d\mu_r=0.$$

Similarly $\int_{X-E}h_n\,d\eta=0$. Q.E.D.

PROOF OF THEOREM 1. If we set $A_0^q=A_0\circ q^{-1}=\{f\circ q^{-1}:f\in A_0\}$, then $A_0(q)=A_0\cap A_0^q$ where, as observed in [3], A_0 and A_0^q satisfy the hypothesis of Lemma 1. Since every closed ideal of A_0^q has the form $I_2\circ q^{-1}$, where I_2 is a closed ideal of A_0, Theorem 1 follows.

Suppose next that A is a closed subalgebra of $C(X)$ containing 1 and that G is a group of homeomorphisms of X onto itself, such that $f\circ g\in A$ for every $f\in A$, $g\in G$. For $S\subseteq C(X)$ set $S_G=\{f\in S:f\circ g=f$ for every $g\in G\}$.

LEMMA 2. *Assume G is finite. Then if J is a closed ideal of A, J_G is a closed ideal of A_G and every closed ideal of A_G arises in this way.*

PROOF. Let I be a closed ideal of A_G. We will prove that

$$I = [\mathrm{Cl}(A \cdot I)]_G. \tag{1}$$

We show first

$$[\mathrm{Cl}(A \cdot I)]_G = \mathrm{Cl}[(A \cdot I)_G]. \tag{2}$$

For suppose that $f \in [\mathrm{Cl}(A \cdot I)]_G$. Then there exists a sequence $\{f_n\}$, $f_n \in A \cdot I$, which converges uniformly to f. Let N be equal to the order of G. Then, if $F_n = N^{-1} \sum_{g \in G} f_n \circ g$, $F_n \in (A \cdot I)_G$ and $F_n \to N^{-1} \sum_{g \in G} f \circ g = f$. Hence $f \in \mathrm{Cl}[(A \cdot I)_G]$, proving (2).

Since $I \subseteq [\mathrm{Cl}(A \cdot I)]_G$, in order to prove (1) it suffices to prove that $I^\perp \subseteq [\mathrm{Cl}(A \cdot I)]_G^\perp$. Thus if $\eta \in I^\perp$, we need only show (by (2)), that η annihilates $(A \cdot I)_G$. Let $h \in (A \cdot I)_G$, where $h = \sum_{r=1}^{p} a_r i_r$, $a_r \in A$, $i_r \in I$. Since $i_r \eta \in A_G^\perp$ we have

$$\begin{aligned} \int h \, d\eta &= \frac{1}{N} \int \left(\sum_{g \in G} h \circ g \right) d\eta = \frac{1}{N} \int \sum_{g \in G} \sum_{r=1}^{p} (a_r \circ g) \cdot i_r \, d\eta \\ &= \frac{1}{N} \sum_{r=1}^{p} \int \left(\sum_{g \in G} a_r \circ g \right) \cdot i_r \, d\eta = 0. \end{aligned}$$

Q.E.D.

PROOF OF THEOREM 2. Let G' be the two element group of homeomorphisms of T generated by q (so that $A_q = (A_0)_{G'}$). If I is a closed ideal of A_q, we obtain by means of Lemma 2 and Theorem 1

$$\begin{aligned} I &= (I[A_0(q)])_{G'} = (I[A_0] \cap I[A_0^q])_{G'} = (I[A_0])_{G'} \cap (I[A_0^q])_{G'} \\ &= (I[A_0])_{G'} = A_q \cap I[A_0]. \end{aligned}$$

Q.E.D.

PROOF OF THEOREM 3. Define $p_1 : T \cup T' \to T \cup T'$ by $p_1 | T = p$, $p_1 | T' = p^{-1}$. Then $p_1 \circ p_1 = \text{identity}$, $T \cup T' / p_1$ may be identified with T, and if G'' denotes the two element group of homeomorphisms of $T \cup T'$ on itself generated by p_1, $B_p = B_{G''}$. Thus the proof of Theorem 3 is analogous to the proof of Theorem 2. Q.E.D.

BIBLIOGRAPHY

1. R. G. Blumenthal, *Maximality in function algebras*, Canad. J. Math. **22** (1970), 1002–1004.

2. A. G. Brandstein, *A class of hypo-Dirichlet algebras*, Proc. Amer. Math. Soc. **28** (1971), 501–504.

3. A. Browder and J. Wermer, *A method for constructing Dirichlet algebras*, Proc. Amer. Math. Soc. **15** (1964), 546–552. MR **29** #2669.

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