SEMIGROUPS CORRESPONDING TO ALGEBROID BRANCHES IN THE PLANE

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Abstract. The symmetric semigroups of nonnegative integers and their generators, corresponding to algebroid branches of the plane, are determined.

Let \( \alpha \) be an algebroid branch of a plane curve with coefficients in an algebraically closed field with characteristic 0. Although the semigroup \( S(\alpha) \) of \( \alpha \) is symmetric, not all symmetric semigroups correspond to branches [1]. Subsequently the nonredundant generators of \( S(\alpha) \) are found from which follow necessary and sufficient conditions for a nonredundant set of positive integers to generate the semigroup of a branch \( \alpha \). A method for obtaining the generators from the power series is given.

The definitions for infinitely near points \( P_j \), multiplicity sequence, proximity structure, satellite cluster and multiplicity matrix are given in [3]. By the restriction of a point \( P_j \) we mean the number of points \( P_j \) is proximate to, and the leading points are the points which have successors with increased restriction. Defining the order of a divisor \( D \) on \( \alpha, o(D, \alpha) \), in the usual way, let \( v(P_j, \alpha) = \min(o(D, \alpha)) \), where \( \alpha \) passes thru \( P_j \) and \( D \) is a divisor of \( P_j \).

Lemma 1. If \( \alpha \) has satellite clusters \( S_1, \ldots, S_n \) and \( \alpha^* \) is obtained from \( \alpha \) by deleting \( S_n \), then \( v(P_j, \alpha^*) = v(P_j, \alpha) \), \( 0 \leq j \leq l \), \( P_l \) the last satellite point of \( \alpha^* \).

Proof. Let \( M(\alpha) = (m_{ij}) \), \( 0 \leq j \leq l \), be the upper triangular multiplicity matrix of \( \alpha \). Then \( v(P_j, \alpha) = \sum_{i=0}^{l} m_{ij} m_{ij} \), \( 0 \leq j \leq l \) [2]. Since \( M(\alpha^*) = (m_{ij}^*) \) consists of the first \( l \) rows and columns of \( M(\alpha) \), by using the proximity structure, for \( j \leq l \),

\[
\sum_{i=0}^{l} m_{ij} m_{ij} = \sum_{i=0}^{l} m_{ij}^* (m_{ij} m_{ij}) = m_{ij} v(P_j, \alpha^*).
\]

Lemma 2. If \( L_j \) are the leading points, \( 0 \leq j \leq n \), then

\[
g.c.d. (v(L_0, \alpha), \ldots, v(L_n, \alpha)) = d = 1.
\]

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Proof. If \( \alpha \) has a single satellite cluster, then \( \text{g.c.d.} (v(L_0, \alpha), v(L_1, \alpha)) = m_{tt} = 1 \). Let \( \alpha \) have \( n \) clusters. Assuming the truth for \( \alpha^* \), defined in Lemma 1, implies \( d|m_{tt} \). Let \( L_{n+1} = P_{r+1} \). Since \( v(P_{r+1}, \alpha) = \sum_{i=0}^{r} m_{ir} \cdot m_{tt} + m_{r+1,t} \) and since \( m_{tt}|m_{r+1,t} \), \( 1 \leq i \leq r \), \( d|m_{r+1,t} \Rightarrow d|\text{g.c.d.} (m_{tt}, m_{r+1,t}) = 1 \).

**Corollary 1.** Let \( \alpha_j \) be the branch whose only satellite cluster is \( S_j \), \( 1 \leq j \leq n \), with multiplicity \( \lambda_j \) at the origin and let \( \lambda_{n+1} = 1 \). Then \( v(L_0, \alpha) = \prod_{i=1}^{n+1} \lambda_i \) and \( \text{g.c.d.} (v(L_0, \alpha), \cdots, v(L_j, \alpha)) = \prod_{i=1}^{n+1} \lambda_i \).

**Lemma 3.** Let \( \alpha \) have a single satellite cluster. Let \( L_1 = P_{r+1} \). Then

\[
m_{rt}m_{r+1,t} = \sum_{t=r+1}^{t} (m_{it})^2.
\]

Proof.

\[
m_{rt} = q \cdot m_{r+1,t} + m_{r+1+q,t}.
\]

Rightarrow \( m_{rt}m_{r+1,t} = \sum_{t=r+1}^{t} (m_{it})^2 + m_{r+1,t}m_{r+1+q,t} \).

Repeating the process with \( m_{r+1,t} \) and \( m_{r+1+q,t} \) proves Lemma 3.

**Lemma 4.** Let \( E_j \) denote the last satellite point of the \( j \)th cluster of a branch \( \alpha \) with \( n \) clusters. Then \( v(E_j, \alpha) = \lambda_j v(L_j, \alpha) \).

Proof. Assume \( j = n \) and \( L_n = P_{r+1} \). If \( P_q \) is proximate to \( P_{q-k} \) and \( P_{q+k} \), then \( m_q = 1 \) and \( m_q = m_{s,q-1} + m_{s,q+k} \), \( s < q \). Therefore the proximity structure and Lemma 3 imply

\[
v(E_n, \alpha) = (m_{rt} - m_{r+1,t})v(P_r, \alpha) + m_{r+1,t}v(P_{r+1}, \alpha) + \sum_{i=r+2}^{t} (m_{it})^2
\]

\[
= (m_{rt} - m_{r+1,t})(v(P_r, \alpha) + m_{r+1,t}) + m_{r+1,t}v(P_{r+1}, \alpha)
\]

\[
= \lambda_n v(L_n, \alpha).
\]

This and Lemma 1 also prove Lemma 4, if \( j \neq n \).

**Corollary 2.** If \( n \geq 2 \), then \( \sum_{j=1}^{n-1} (\lambda_j - 1) v(L_j, \alpha) < v(L_n, \alpha) \).

Proof. Induction on \( n \).

**Lemma 5.** If \( \sum_{j=1}^{n-1} a_j v(L_j, \alpha) = \sum_{j=0}^{n} b_j v(L_j, \alpha) \), \( a_j \) and \( b_j \) nonnegative integers, and if \( 0 \leq a_j < \lambda_j \), then \( b_0 = 0 \) and \( a_j = b_j \), \( 1 \leq j \leq n \).

Proof. For one satellite cluster the statement follows from \( \text{g.c.d.} (v(L_0, \alpha), v(L_1, \alpha)) = 1 \). Assume \( \alpha \) has \( n \geq 2 \) clusters and assume two different representations

\[
\sum_{j=1}^{n-1} a_j v(L_j, \alpha) + a_n v(L_n, \alpha) = \sum_{j=0}^{n-1} b_j v(L_j, \alpha) + b_n v(L_n, \alpha).
\]
By Corollary 2, \( b_n \leq a_n \Rightarrow a_n = b_n \), since \( \lambda_n | a_n - b_n \Rightarrow \sum_{j=0}^{n-1} a_j (v(L_j, \alpha) / \lambda_n) = \sum_{j=0}^{n-1} b_j (v(L_j, \alpha) / \lambda_n) \), a contradiction if the truth is assumed for \( n-1 \) clusters.

**Theorem 1.** \( S(\alpha) \) is generated nonredundantly by \( v(L_i, \alpha), 0 \leq j \leq n \).

**Proof.** By Lemma 5, \( \sum_{j=1}^{n} a_j v(L_j, \alpha), 0 \leq a_j < \lambda_j \), are distinct first elements of \( S(\alpha) \) in the congruence classes \( \text{mod}(v(L_0, \alpha)), \prod_{j=1}^{n} \lambda_j = v(L_0, \alpha) \) in number.

Let \( S=\{a_0 < a_1 < \cdots < a_n\} \) be a set of nonredundant integers, \( n \geq 1 \).

Let \( \lambda_i = \gcd(a_0, \cdots, a_i) = 1 \) and recursively

\[
\lambda_j = \gcd(a_j \mid \prod_{i=j+1}^{n+1} \lambda_i), \quad 2 \leq j \leq n,
\]

\[
\prod_{i=j+1}^{n+1} \lambda_i, \quad 1 \leq j \leq n.
\]

Let \( \lambda_i = a_0 \prod_{i=2}^{i=i} \lambda_i \). From Lemmas 1 and 4 then follows

**Theorem 2.** \( S \) generates a semigroup \( S(\alpha) \) iff \( \lambda_j > 1, 2 \leq j \leq n \), and \( \lambda_j a_j < a_{j+1}, 1 \leq j \leq n-1 \).

Let \( \alpha \) have the representative

\[
x_\alpha = u^B, \quad y_\alpha = \sum_{h=1}^{g(0)} a_{h,0} u^{h \gamma_0} + \sum_{h=1}^{g(1)} a_{h,1} u^{h \gamma_1 + (h-1) d_1} + \cdots + \sum_{h=1}^{g(n-1)} a_{h,n-1} u^{h \gamma_{n-1} + (h-1) d_{n-1}} + a_{1,n} u^\gamma + \sum_{h=1}^{\infty} a_{h} u^{n+h},
\]

where \( a_{0,i} \neq 0, 1 \leq j \leq n, 1 < d_j = \gcd(v_0, \cdots, v_j), 1 \leq j \leq n-1, 1 = d_n = \gcd(v_0, \cdots, v_n) \).

Then \( d_j = \prod_{i=j+1}^{i=n+1} \lambda_i, \quad v_0 = \prod_{i=1}^{i=n+1} \lambda_i, \quad \lambda_i \) defined in Corollary 1, \( v_j = v(L_j, \alpha) \) and for \( 2 \leq j \leq n, v_j = m(j) + v_{j-1} + k_{j-1} d_j \), where \( m(j) \) is the multiplicity of \( L_j \) and \( k_{j-1} + 1 \) is the number of free points following the \( (j-1) \)st cluster [3, Chapter XI, \( 5.6.1 \)]. Hence \( v_j = v(L_1, \alpha) + \sum_{i=2}^{i=m} (m(i) + k_{i-1} d_i) \). By Lemma 4,

\[
\lambda_{i-1} v(L_{i-1}, \alpha) + k_{i-1} d_i + m(i) = v(L_i, \alpha)
\]

\[
\Rightarrow v_j = \sum_{i=1}^{i=j} v(L_i, \alpha) - \sum_{i=2}^{i=j} \lambda_{i-1} v(L_{i-1}, \alpha)
\]

\[
\Rightarrow v(L_j, \alpha) = v_j + \sum_{i=2}^{i=j} (\lambda_{i-1} - 1) v(L_{i-1}, \alpha),
\]

which determines the generators recursively.
REFERENCES


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