A CONVERGENCE THEOREM FOR LIMITÄRPERIODISCH T-FRACTIONS OF RATIONAL FUNCTIONS

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Abstract. We prove that a limitärpériodisch T-fraction, which corresponds to a rational function, converges (locally uniformly) to the original function in a certain domain.

1. Introduction. The sequence \(\{A_n(z)/B_n(z)\}\) obtained by the rule

\[
\frac{A_n(z)}{B_n(z)} = 1 + d_0 z + \frac{z}{1 + d_1 z + \cdots + 1 + d_n z}
\]

is called a T-fraction (see [5]). Moreover, the T-fraction is called limitärpériodisch if the sequence \(\{d_n\}\) converges.

The T-fraction is said to converge for a certain \(z\)-value, if for that particular value

\[
\lim_{n \to \infty} \left( 1 + d_0 z + \frac{z}{1 + d_1 z + \cdots + 1 + d_n z} \right)
\]

exists in \(C\).

The T-fraction is said to correspond to the power series (*) \(1 + \sum_{n=1}^{\infty} c_n z^n\) if (*) agrees with the power series expansion of \(A_n(z)/B_n(z)\) up to and including the term \(c_{k(n)} z^{k(n)}\), where \(k(n)\to\infty\) as \(n\to\infty\).

For every formal power series \(1 + \sum_{n=1}^{\infty} c_n z^n\), and thus for every function \(f_0\), holomorphic in some region containing the origin, and normalized by \(f_0(0)=1\), there is exactly one corresponding T-fraction.

(A proof is given in [5].)

Starting with the function \(f_0\), we obtain the T-fraction expansion in the following way:

Let \(\{f_n\}\) be the sequence of functions, defined by

\[
f_n(z) = 1 + (f_n'(0) - 1)z + \frac{z}{f_{n+1}(z)}, \quad z \neq 0, n = 0, 1, 2, \cdots,
\]

\[
f_{n+1}(0) = 1.
\]
With
\[ d_n = f'_n(0) - 1, \quad n = 0, 1, 2, \ldots \]
the continued fraction (1) is the \( T \)-fraction of \( f_0 \).

Due to the linearity of the elements of the \( T \)-fraction a great deal can be said about the convergence. Several convergence theorems are proved in [2], [3], and [5]. The criteria are given in terms of conditions on the sequence \( \{d_n\} \). A different kind of result is proved in [6], where convergence properties of the \( T \)-fraction expansion is concluded from boundedness conditions of the function. A step in the proof is to establish the following lemma (see [6, p. 8]):

**Lemma 1.** Let \( f_0 \) be holomorphic in \( |z| < 1 \), normalized by \( f_0(0) = 1 \), and such that the function \( f_n \), defined in (2) all are holomorphic in \( |z| < 1 \). Further, let \( f_0 \) have a \( T \)-fraction expansion where \( d_n \to -1 \) as \( n \to \infty \). Then the \( T \)-fraction of \( f_0 \) converges to \( f_0 \) uniformly on any compact subset of the open unit disk.

For rational \( f_0 \) this result can be extended in the following way (announced in [1]):

2. **The main result.**

**Theorem 1.** Let \( f_0 \) be a rational function normalized by \( f_0(0) = 1 \) and with \( \text{limitärperiodisch} \) \( T \)-fraction. Take an arbitrary \( \theta \in (0, 1) \) and let \( D_\theta \) denote the disk \( \{z; |z| \leq \theta\} \). Remove from \( D_\theta \) arbitrary neighborhoods of the poles of \( f_0 \) in \( D_\theta \). Then the \( T \)-fraction of \( f_0 \) converges to \( f_0 \) uniformly on the remaining set \( D_\theta^* \).

**Remark 1.** In this theorem the interval \( (0, 1) \) cannot be replaced by \( (0, r) \) where \( r \geq 1 \), as may be seen from the classic example \( f_0 = 1 \). (The \( T \)-fraction of this function has the form
\[
1 - z + \frac{z}{1 - z + \cdots + \frac{z}{1 - z + \cdots}}
\]
and converges in \( |z| < 1 \) to 1, in \( |z| > 1 \) to \(-z\) and diverges on the unit circle, except for \( z = -1 \), where it converges to 1).

**Remark 2.** The existence of an uncountable set of rational functions with nontrivial \( \text{limitärperiodisch} \) \( T \)-fractions is proved in [1]. Applying the functions used in this proof we can prove the existence of (an uncountable set of) rational functions with poles in \( |z| < 1 \) and with nontrivial \( \text{limitärperiodisch} \) \( T \)-fractions.

For such functions the \( T \)-fraction expansion converges in a larger domain than the power series expansion. To prove Theorem 1 we state some
3. Preliminary results. From now on we consider a normalized rational function \( f_0 \), i.e. let \( f_0 \) in §1 be given by the formula

\[
f_0(z) = \frac{1 + \sum_{k=1}^{m_0} \beta_k^{(-1)} z^k}{1 + \sum_{k=1}^{m_0} \beta_k^{(0)} z^k},
\]

where \( \beta_k^{(-1)} \), \( \beta_k^{(0)} \) are arbitrary (complex) constants, and let \( \{f_n\} \) and \( \{d_n\} \) be the sequences defined in (2) and (3) respectively. Then, for \( n=1, 2, 3, \cdots \), we have

\[
f_n(z) = \frac{1 + \sum_{k=1}^{m_0} \beta_k^{(n-1)} z^k}{1 + \sum_{k=1}^{m_0} \beta_k^{(n)} z^k},
\]

where the constants \( \beta_k^{(n)} \) are given by certain recursion formulas (see [1]).

Furthermore we shall need some well-known recursion formulas from the theory of continued fractions. Specializing to the present case and using the notation from §1, we have

\[
A_n(z)B_{n-1}(z) - A_{n-1}(z)B_n(z) = (-1)^{n-1} z^n,
\]

\[
A_m(z)/B_m(z)
\]

\[
= \left( A_{n-1}(z) \left[ 1 + d_n z + \frac{z}{1 + d_n z + \cdots + 1 + d_m z} + z A_{n-2}(z) \right] \right)
\]

\[
\cdot \left( B_{n-1}(z) \left[ 1 + d_n z + \frac{z}{1 + d_n z + \cdots + 1 + d_m z} + z B_{n-2}(z) \right] \right)^{-1}.
\]

where \( A_k(z) \) and \( B_k(z) \) are polynomials, given by the recursion formulas

\[
A_{-1}(z) = 1, \quad B_{-1}(z) = 0, \quad A_0(z) = 1 + d_0 z, \quad B_0(z) = 1,
\]

\[
A_n(z) = (1 + d_n z)A_{n-1}(z) + z A_{n-2}(z), \quad B_n(z) = (1 + d_n z)B_{n-1}(z) + z B_{n-2}(z), \quad n = 1, 2, 3, \cdots .
\]

Immediately from (1), (2), and (8) it follows inductively

\[
A_{n-1}(z)f_n(z) + z A_{n-2}(z) = f_0(z)f_1(z) \cdots f_n(z),
\]

\[
B_{n-1}(z)f_n(z) + z B_{n-2}(z) = f_1(z)f_2(z) \cdots f_n(z),
\]

and in particular,

\[
f_0(z) = \frac{A_{n-1}(z)f_n(z) + z A_{n-2}(z)}{B_{n-1}(z)f_n(z) + z B_{n-2}(z)}.
\]

Finally we rephrase Theorem 2.42 in [4] as

**Theorem 2.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of complex-valued functions of the complex variable \( z \), defined in some region \( R \), and assume
that they converge uniformly in $R$ to limit functions $a$ and $b$ respectively. Assume further the existence of a positive $\delta < 1$ and two positive numbers $c$ and $C$, such that, in the whole region $R$, $c \leq |p_1| \leq C$, $|p_2| p_1 \leq \delta$, where $p_1$ and $p_2$ denote the roots of the quadratic equation $p^2 - b p - a = 0$. Then there is an $N$, such that for $n \geq N$ the continued fraction

$$
\frac{a_{n+1}(z)}{b_{n+1}(z)} + \frac{a_{n+2}(z)}{b_{n+2}(z)} + \cdots
$$

converges uniformly in $R$ to a finite-valued limit function.

**Proof of Theorem 1.** (The first part of the proof is almost identical to the first part of Waadeland’s proof of Lemma 1, while the second part strongly depends on the present conditions.)

Let

$$
1 + d_0 z + \frac{z}{1 + d_1 z + \cdots + 1 + d_n z + \cdots}
$$

be the $T$-fraction of $f_0$. By the hypothesis $\{d_n\}$ is convergent, and from Theorem 1 of [1] we know $d_n \to -1$ as $n \to \infty$. Therefore, putting $a_n(z) = z$, $b_n(z) = 1 + d_n z$, and $R = \{z; |z| < \theta\}$ with $\theta \in (\theta, 1)$, we see that the convergence conditions of Theorem 2 are satisfied. Furthermore the inequalities are obviously valid since $p_1(z) = 1$ and $p_2(z) = -z$ in the present case. Thus we conclude that there exists a number $N$ such that for $n \geq N$ the $T$-fraction of $f_n$,

$$
1 + d_n z + \frac{z}{1 + d_{n+1} z + 1 + d_{n+2} z + \cdots},
$$

converges uniformly on $D_\theta$ to a limit function $g$ (finite-valued). We assert that $g = f_n$ (restricted to $D_\theta$). Proof of this: The uniform convergence of the approximants of

$$
1 + d_n z + \frac{z}{1 + d_{n+1} z + 1 + d_{n+2} z + \cdots}
$$

implies, by local considerations, the continuity of $g$ (finite-valued). In particular $g$ is bounded on $D_\theta$, which in turn implies uniform boundedness of the sequence of approximants and thus regularity of

$$
1 + d_n z + \frac{z}{1 + d_{n+1} z + \cdots + 1 + d_m z}
$$
on $D_\theta$ for all $m \geq M$ for some $M$. By Weierstrass we conclude that $g$ is holomorphic in $|z| < \theta$ and, for $k = 1, 2, 3, \cdots$,

$$
g^{(k)}(0) = \lim_{m \to \infty} \frac{d^k}{dz^k} \left[ 1 + d_n z + \frac{z}{1 + d_{n+1} z + \cdots + 1 + d_m z} \right]_{z=0}.
$$
On the other hand, from the correspondence between \( f_n(z) \) and (12) we have
\[
\lim_{m \to \infty} \frac{1}{d_k} \left[ 1 + d_n z + \frac{z}{1 + d_{n+1} z + \cdots + 1 + d_m z} \right] = f_n^{(k)}(0),
\]
\( k = 1, 2, 3, \ldots \) (see Theorem 2.1 in [5]).

This agreement in Maclaurin series expansion of \( f_n \) and \( g \) shows that \( f_n \) agrees with \( g \) on \( D_\theta \).

To finish the proof of the theorem, fix \( n \geq N \) and consider \( m \geq M \). We shall find it convenient to define functions \( r_m \) on \( D_\theta \) given by the formulas
\[
r_m(z) = 1 + d_n z + \frac{z}{1 + d_{n+1} z + \cdots + 1 + d_m z} - f_n(z).
\]
Thus, from (7) and (11) the following holds in \( D_\theta^* \):
\[
\begin{align*}
|f_0(z) - A_m(z)/B_m(z)| &= \left| (z r_m(z)(A_{n-2}(z)B_{n-1}(z) - A_{n-1}(z)B_{n-2}(z)) \right| \\
&= \left| (B_{n-1}(z) f_n(z) + z B_{n-2}(z))(B_{n-1}(z) (f_n(z) + r_m(z)) + z B_{n-2}(z))^{-1} \right|.
\end{align*}
\]

Applying (6), we finally get
\[
|f_0(z) - A_m(z)/B_m(z)| = (|r_m(z)| |z^n|)
\cdot \left| (B_{n-1}(z) f_n(z) + z B_{n-2}(z)| B_{n-1}(z)(f_n(z) + r_m(z)) + z B_{n-2}(z))^{-1} \right|.
\]

We know that \( r_m \) converges to 0 uniformly on \( D_\theta \). Since \( B_{n-1}, B_{n-2}, \) and \( f_n \) are holomorphic in \( D_\theta \), we are done if we can show that \( G \) defined by
\[
G(z) = B_{n-1}(z) f_n(z) + z B_{n-2}(z)
\]
has all its zeros among the poles of \( f_0 \). This, however, follows easily if we combine (5) and (10):
\[
B_{n-1}(z) f_n(z) + z B_{n-2}(z) = \frac{1 + \sum_{k=1}^{m} \beta_k^{(0)} z^k}{1 + \sum_{k=1}^{m} \beta_k^{(n)} z^k}.
\]

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