PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

# A NOTE ON HOMOTOPY EQUIVALENCES

R. M. VOGT

ABSTRACT. Given a homotopy equivalence $f: X \to Y$, a homotopy inverse $g$ of $f$, and a homotopy $H: X \times I \to X$ from $g \circ f$ to $1_X$. We show that there is a homotopy $K: Y \times I \to Y$ from $f \circ g$ to $1_Y$ such that $f \circ H \simeq K \circ (f \times 1_I)$ rel $X \times \partial I$ and $H \circ (g \times 1_I) \simeq g \circ K$ rel $Y \times \partial I$.

In [1], R. Lashof introduced the notion of a strong homotopy equivalence in context with the study of reductions of topological microbundles to piecewise linear or differentiable microbundles.

DEFINITION. A strong homotopy equivalence between two spaces $X$ and $Y$ is a quadruple $(f, g, H, K)$ where $f: X \to Y$ and $g: Y \to X$ are maps and $H: X \times I \to X$ and $K: Y \times I \to Y$ are homotopies, $H: g \circ f \simeq 1_X$, $K: f \circ g \simeq 1_Y$, such that $f \circ H \simeq K \circ (f \times 1_I)$ rel endpoints and $H \circ (g \times 1_I) \simeq g \circ K$ rel endpoints (here $I$ denotes the unit interval).

Now the question arises, whether any homotopy equivalence can be made into a strong one. The following proposition gives an affirmative answer. The result is also quite useful for many questions in homotopy theory. Although it is implicitly contained in various papers, I have never found it stated explicitly. The proof is very elementary, only using the basic facts about track addition of homotopies.

PROPOSITION. *Let $f: X \to Y$ be a homotopy equivalence with homotopy inverse $g$. Let $H: X \times I \to X$ be a homotopy from $g \circ f$ to $1_X$. Then there exists a homotopy $K: Y \times I \to Y$ from $f \circ g$ to $1_Y$ such that $f \circ H \simeq K \circ (f \times 1_I)$ rel endpoints and $H \circ (g \times 1_I) \simeq g \circ K$ rel endpoints.*

Let $[X, Z]_p^q$ denote the set of equivalence classes of homotopies $F: X \times I \to Z$ such that $F|X \times 0 = p: X \to Z$ and $F|X \times 1 = q: X \to Z$. Two such homotopies are defined to be equivalent if they are homotopic rel endpoints.

Maps $h: X \to Y$ and $k: Z \to W$ induce maps $h^*: [Y, Z]_p^q \to [X, Z]_{p \circ h}^{q \circ h}$, given by $h^*[M] = [M \circ (h \times 1)]$, where $[M]$ denotes the equivalence class of $M$, and $k_*: [X, Z]_u^v \to [X, W]_{k \circ u}^{k \circ v}$, given by $k_*[N] = [k \circ N]$.

LEMMA 1. *Let $[R] \in [U, V]_k^l$ and $[Q] \in [V, W]_u^v$. Then*

$$[u \circ R + Q \circ (l \times 1)] = [Q \circ (k \times 1) + v \circ R]$$

Received by the editors August 19, 1970.
*AMS* 1970 *subject classifications*. Primary 55D10.

*in* $[U, W]^{v\circ l}_{u\circ k}$, *where* "+" *denotes track addition, the left homotopy is applied first.*

PROOF. The map $G: U\times I\times I\to W$, given by $G(x, t_1, t_2)=Q(R(x, t_1), t_2)$ is the required homotopy rel endpoints between $u\circ R+Q\circ(l\times 1)$ and $Q\circ(k\times 1)+v\circ R$, because $G|U\times 0\times I=Q\circ(k\times 1)$, $g|U\times 1\times I=Q\circ(l\times 1)$, $g|U\times I\times 0=u\circ R$, $G|U\times I\times 1=v\circ R$.

LEMMA 2. *With the notation of the proposition,* $f^*: [Y, Z]^q_p\to[X, Z]^{q\circ f}_{p\circ f}$ *and* $g^*: [X, Z]^v_u\to[Y, Z]^{v\circ g}_{u\circ g}$ *are bijections.*

PROOF. Consider the composite

$$(*)\qquad r: [Y, Z]^q_p \xrightarrow{f^*} [X, Z]^{q\circ f}_{p\circ f} \xrightarrow{g^*} [Y, Z]^{q\circ f\circ g}_{p\circ f\circ g} \xrightarrow{s} [Y, Z]^q_q$$

where $s$ is given by $s[M]=[-p\circ F+M+q\circ F]$, and $F: f\circ g\simeq 1_Y$ is an arbitrary fixed homotopy. For $[Q]\in[Y, Z]^q_p$, we have

$$r[Q] = [-p\circ F + Q\circ(f\circ g\times 1) + q\circ F].$$

Substituting $R$ by $F$ in the previous lemma, we find that $r[Q]=[Q]$. Since $s$ is a bijection (with inverse $[N]\mapsto[p\circ F+N-q\circ F]$), the composite $g^*\circ f^*$ is a bijection, and hence $f^*$ injective and $g^*$ surjective. Analogously one shows that in the sequence

$$(**)\qquad [X, Z]^v_u \xrightarrow{g^*} [Y, Z]^{v\circ g}_{u\circ g} \xrightarrow{f^*} [X, Z]^{v\circ g\circ f}_{u\circ g\circ f}$$

$g^*$ is injective and $f^*$ is surjective. Hence, in (*), $g^*$ and therefore $f^*$ are bijective. Similarly in (**), $f^*$ and hence $g^*$ are bijective.

PROOF OF THE PROPOSITION. We are given $H: g\circ f\simeq 1_X$. Choose $K$ to be a representative of $f^{*-1}[f\circ H]\in[Y, Y]^1_{f\circ g}$, and let $H'$ be a representative of $g^{*-1}[g\circ K]\in[X, X]^1_{g\circ f}$. Then $K\circ(f\times 1)\simeq f\circ H$ rel endpoints, and $H'\circ(g\times 1)\simeq g\circ K$ rel endpoints. Consider the composite

$$r: [Y, X]^g_{g\circ f\circ g} \xrightarrow{f_*} [Y, Y]^{f\circ g}_{f\circ g\circ f\circ g} \xrightarrow{s} [Y, Y]^1_{f\circ g} \xrightarrow{g_*} [Y, X]^g_{g\circ f\circ g}$$

where $s$ is defined by $s[M]=[-K\circ(f\circ g\times 1)+M+K]$. Let $[R]\in[Y, X]^g_{g\circ f\circ g}$, then

$$\begin{aligned} r[R] &= [-g\circ K\circ(f\circ g\times 1) + g\circ f\circ R + g\circ K] \\ &= [-H'\circ(g\circ f\circ g\times 1) + g\circ f\circ R + H'\circ(g\times 1)] = [R]. \end{aligned}$$

The last equality follows from Lemma 1 by substituting $Q$ by $H'$.

Now $[H\circ(g\times 1)]\in[Y, X]^{g}_{g\circ f\circ g}$, and therefore

$$\begin{aligned}[H\circ(g\times 1)] &= r[H\circ(g\times 1)]\\ &= [-g\circ K\circ(f\circ g\times 1)+g\circ f\circ H\circ(g\times 1)+g\circ K]\\ &= [-g\circ K\circ(f\circ g\times 1)+g\circ K\circ(f\circ g\times 1)+g\circ K]\\ &= [g\circ K].\end{aligned}$$

So $H\circ(g\times 1)\simeq g\circ K$ rel endpoints.

## References

1. R. Lashof, *The immersion approach to triangulation and smoothing*, Proc. Advanced Study Institute on Algebraic Topology, Mathematisk Institut, Aarhus Universitet, Aarhus, 1970.

Matematisk Institut, Aarhus Universitet, Aarhus, Denmark

*Current address*: Mathematisches Institut, Universität des Saarlandes, Saarbrücken, Federal Republic of Germany