A NOTE ON HOMOTOPY EQUIVALENCES

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Abstract. Given a homotopy equivalence $f:X \rightarrow Y$, a homotopy inverse $g$ of $f$, and a homotopy $H:X \times I \rightarrow X$ from $g \circ f$ to $1_X$. We show that there is a homotopy $K: Y \times I \rightarrow Y$ from $f \circ g$ to $1_Y$ such that $f \circ H \simeq K \circ (f \times 1_I)$ rel $X \times \partial I$ and $H \circ (g \times 1_I) \simeq g \circ K$ rel $Y \times \partial I$.

In [1], R. Lashof introduced the notion of a strong homotopy equivalence in context with the study of reductions of topological microbundles to piecewise linear or differentiable microbundles.

Definition. A strong homotopy equivalence between two spaces $X$ and $Y$ is a quadruple $(f, g, H, K)$ where $f: X \rightarrow Y$ and $g: Y \rightarrow X$ are maps and $H: X \times I \rightarrow X$ and $K: Y \times I \rightarrow Y$ are homotopies, $H: g \circ f \simeq 1_X$, $K: f \circ g \simeq 1_Y$, such that $f \circ H \simeq K \circ (f \times 1_I)$ rel endpoints and $H \circ (g \times 1_I) \simeq g \circ K$ rel endpoints (here $I$ denotes the unit interval).

Now the question arises, whether any homotopy equivalence can be made into a strong one. The following proposition gives an affirmative answer. The result is also quite useful for many questions in homotopy theory. Although it is implicitly contained in various papers, I have never found it stated explicitly. The proof is very elementary, only using the basic facts about track addition of homotopies.

Proposition. Let $f: X \rightarrow Y$ be a homotopy equivalence with homotopy inverse $g$. Let $H: X \times I \rightarrow X$ be a homotopy from $g \circ f$ to $1_X$. Then there exists a homotopy $K: Y \times I \rightarrow Y$ from $f \circ g$ to $1_Y$ such that $f \circ H \simeq K \circ (f \times 1_I)$ rel endpoints and $H \circ (g \times 1_I) \simeq g \circ K$ rel endpoints.

Let $[X, Z]_p^q$ denote the set of equivalence classes of homotopies $F: X \times I \rightarrow Z$ such that $F|X \times 0 = p: X \rightarrow Z$ and $F|X \times 1 = q: X \rightarrow Z$. Two such homotopies are defined to be equivalent if they are homotopic rel endpoints.

Maps $h: X \rightarrow Y$ and $k: Z \rightarrow W$ induce maps $h^*: [Y, Z]_p^q \rightarrow [X, Z]_{p \circ h}^{q \circ h}$, given by $h^*[M] = [M \circ (h \times 1)]$, where $[M]$ denotes the equivalence class of $M$, and $k_*: [X, Z]_p^q \rightarrow [X, W]_{k \circ u}^{k \circ w}$, given by $k_*[N] = [k \circ N]$.

Lemma 1. Let $[R] \in [U, V]_p^q$ and $[Q] \in [V, W]_u^w$. Then

\[ [u \circ R + Q \circ (l \times 1)] = [Q \circ (k \times 1) + v \circ R] \]

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in \([U, W]_{\operatorname{vul}}\), where "+" denotes track addition, the left homotopy is applied first.

**Proof.** The map \(G: U \times I \times I \to W\), given by \(G(x, t_1, t_2) = Q(R(x, t_1), t_2)\) is the required homotopy rel endpoints between \(u \circ R + Q \circ (l \times 1)\) and \(Q \circ (k \times 1) + v \circ R\), because \(G|U \times 0 \times I = Q \circ (k \times 1)\), \(g|U \times I \times 0 = u \circ R\), \(G|U \times I \times 1 = v \circ R\).

**Lemma 2.** With the notation of the proposition, \(f^*: [Y, Z]_w \to [X, Z]_{\operatorname{vul}}\) and \(g^*: [X, Z]_w \to [Y, Z]_{\operatorname{vul}}\) are bijections.

**Proof.** Consider the composite

\[
(*) \quad r: [Y, Z]_w^v \xrightarrow{f^*} [X, Z]_{\operatorname{vul}}^v \xrightarrow{g^*} [Y, Z]_{\operatorname{vul}}^v \xrightarrow{s} [Y, Z]_w^v
\]

where \(s\) is given by \(s[M] = [-p \circ F + M + q \circ F]\), and \(F: p \circ g \simeq 1_Y\) is an arbitrary fixed homotopy. For \([Q] \in [Y, Z]_w^v\), we have

\[
r[Q] = [-p \circ F + Q \circ (f \circ g \times 1) + q \circ F].
\]

Substituting \(R\) by \(F\) in the previous lemma, we find that \(r[Q] = [Q]\). Since \(s\) is a bijection (with inverse \([N] \mapsto [p \circ F + N - q \circ F]\)), the composite \(g^* \circ f^*\) is a bijection, and hence \(f^*\) injective and \(g^*\) surjective. Analogously one shows that in the sequence

\[
(**) \quad [X, Z]_w^v \xrightarrow{g^*} [Y, Z]_{\operatorname{vul}}^v \xrightarrow{f^*} [X, Z]_{\operatorname{vul}}^v
\]

\(g^*\) is injective and \(f^*\) is surjective. Hence, in \((*)\), \(g^*\) and therefore \(f^*\) are bijective. Similarly in \((**), f^*\) and hence \(g^*\) are bijective.

**Proof of the Proposition.** We are given \(H: g \circ f \simeq 1_X\). Choose \(K\) to be a representative of \(f^* \circ [f \circ H] \in [Y, Y]_{\operatorname{vul}}\), and let \(H'\) be a representative of \(g^* \circ [g \circ K] \in [X, X]_{\operatorname{vul}}\). Then \(K \circ (f \times 1) \simeq f \circ H\) rel endpoints, and \(H' \circ (g \times 1) \simeq g \circ K\) rel endpoints. Consider the composite

\[
r: [Y, X]_{\operatorname{vul}}^{g} \xrightarrow{f^*} [Y, Y]_{\operatorname{vul}}^{g} \xrightarrow{s} [Y, Y]_{\operatorname{vul}}^{g} \xrightarrow{g^*} [Y, X]_{\operatorname{vul}}^{g}
\]

where \(s\) is defined by \(s[M] = [-K \circ (f \circ g \times 1) + M + K]\). Let \([R] \in [Y, X]_{\operatorname{vul}}^{g}\), then

\[
r[R] = [-g \circ K \circ (f \circ g \times 1) + g \circ f \circ R + g \circ K]
\]

\[
= [-H' \circ (g \circ f \circ g \times 1) + g \circ f \circ R + H' \circ (g \times 1)] = [R].
\]

The last equality follows from Lemma 1 by substituting \(Q\) by \(H'\).
Now \([H \circ (g \times 1)] \in [Y, X]_{\partial S^1}^g\), and therefore

\[
[H \circ (g \times 1)] = r[H \circ (g \times 1)]
= [-g \circ K \circ (f \circ g \times 1) + g \circ f \circ H \circ (g \times 1) + g \circ K]
= [-g \circ K \circ (f \circ g \times 1) + g \circ K \circ (f \circ g \times 1) + g \circ K]
= [g \circ K].
\]

So \(H \circ (g \times 1) \simeq g \circ K\) rel endpoints.

REFERENCES


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