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DARBOUX'S THEOREM FAILS FOR WEAK SYMPLECTIC FORMS

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ABSTRACT. An example of a weak symplectic form on a Hilbert space for which Darboux's theorem fails is given.

Introduction. Let E be a Banach space and $B: E\times E\to R$ a continuous bilinear form. Let $B^\flat: E\to E^*$ be defined by $B^\flat(e)\cdot f=B(e,f)$. Call B *nondegenerate* if $B^\flat$ is an isomorphism and call B *weakly nondegenerate* if $B^\flat$ is injective. For a symmetric bilinear form G on E, define the skew form $\tilde{G}$ on $E\times E$ by

$$\tilde{G}((e_1, e_2), (f_1, f_2)) = G(f_2, e_1) - G(e_2, f_1).$$

It is easily seen that $\tilde{G}$ is nondegenerate (resp. weakly nondegenerate) iff G is.

Now let M be a Banach manifold. A *symplectic form* (resp. *weak symplectic form*) on M is a smooth closed two form ω on M such that for each $p\in M$, ω as a bilinear form on T_pM is nondegenerate (resp. weakly nondegenerate); here T_pM is the tangent space at p. Using a technique of Moser, Weinstein ([6], [7]) showed that for each $p\in M$ there is a local chart about p on which ω is constant. This is a significant generalization and simplification of the classical theorem of Darboux. However, in many physical examples (the wave equation and fluid mechanics for instance) one deals with weak symplectic forms (see [1], [3], [4], [5]).

It is therefore interesting to know if Darboux's theorem remains valid for weak symplectic forms. In this note we give a counterexample.

Sympletic forms induced by metrics. If M is a manifold, its cotangent bundle T^*M carries a canonical symplectic form ω. If M is modeled on a reflexive space the form is nondegenerate; otherwise it is only weakly nondegenerate. See [1], [4]. Now let $\langle\ ,\ \rangle_p$ be a (smooth) weak riemannian metric on M. Then it induces a map of TM to T^*M. The pull back Ω of ω to TM is called the form *induced by the metric*. It is a weak symplectic

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form and in a chart U for M it is given by (using principal parts):

$$2\Omega_{u,e}((e_1, e_2), (e_3, e_4))$$
$$= D_u\langle e, e_1\rangle_u \cdot e_3 - D_u\langle e, e_3\rangle_u \cdot e_1 + \langle e_4, e_1\rangle_u - \langle e_2, e_3\rangle_u.$$

Here, D_u denotes the derivative of the map $u \mapsto \langle e, e_1\rangle_u$ with respect to u. In the finite dimensional case this corresponds to the classical formula

$$\Omega = \sum g_{ij}\, dq^i \wedge d\dot{q}^j + \sum \frac{\partial g_{ij}}{\partial q^k}\, \dot{q}^i\, dq^j \wedge dq^k.$$

Observe that in the finite dimensional case if we take new variables $q^1, \cdots, q^n, p_1, \cdots, p_n$ where $p_i = \sum g_{ij}\dot{q}^j$, then (as is easy to check) $\Omega = \sum dq^i \wedge dp_i$ which gives a chart in which Ω is constant.

The example. The following is a simplification of an earlier example. We thank the referee and Paul Chernoff for suggestions in this regard.

Let $\boldsymbol{H}$ be a real Hilbert space. Let $S: \boldsymbol{H} \to \boldsymbol{H}$ be a compact operator with range a dense, but proper subset of $\boldsymbol{H}$, which is selfadjoint and positive: $\langle Sx, x\rangle > 0$ for $0 \neq x \in \boldsymbol{H}$. For example if $\boldsymbol{H} = L_2(\boldsymbol{R})$, we can let $S = (1-\Delta)^{-1}$ where Δ is the Laplacian; the range of S is $H^2(\boldsymbol{R})$.

Since S is positive, -1 is clearly not an eigenvalue. Thus, by the Fredholm alternative, $aI + S$ is onto for any real scalar $a > 0$. Define on $\boldsymbol{H}$ the weak metric $g(x)(e, f) = \langle A_x e, f\rangle$ where $A_x = S + \|x\|^2 I$. Clearly g is smooth in x, and is an inner product. Let Ω be the weak symplectic form on $\boldsymbol{H} \times \boldsymbol{H} = \boldsymbol{H}_1$ induced by g, as was discussed above.

PROPOSITION. *There is no coordinate chart about* $(0, 0) \in \boldsymbol{H}_1$ *on which* Ω *is constant.*

PROOF. If there were such a chart, say $\phi: U \to \boldsymbol{H} \times \boldsymbol{H}$ where U is a neighborhood of $(0, 0)$, then in particular in this chart, the range F of $\Omega^\flat$, as a map of $\boldsymbol{H}_1$ to $\boldsymbol{H}_1^*$, would be constant. Let $B_{x,y}$ be the derivative of ϕ at $(x, y) \in \boldsymbol{H}_1$. Then we obtain that the range of $\Omega^\flat_{x,y}$ equals $B^*_{x,y}F$.

Now by the above formula for Ω, at the point $(x, 0)$ we have

$$2\Omega_{(x,0)}((e_1, e_2), (e_3, e_4)) = g_x(e_4, e_1) - g_x(e_2, e_3).$$

But by construction, for $x \neq 0$, g_x is a strong metric (i.e., A_x is onto for $x \neq 0$), so the range of $\Omega^\flat_{(x,0)}$ is all of $\boldsymbol{H}_1^*$ for $x \neq 0$. Since $B_{x,y}$ is an isomorphism, this implies that $\Omega^\flat_{(0,0)}$ is onto all of $\boldsymbol{H}_1^*$ as well. But g_0 is only a weak metric which is not onto as a map of $\boldsymbol{H}_1$ to $\boldsymbol{H}_1^*$. Hence $\Omega^\flat_{(0,0)}$ cannot be onto as well, a contradiction.

As was pointed out by the referee, the example even shows that Ω cannot be made constant on a continuous vector bundle chart on $T^2M \to TM$, let alone by a manifold chart on TM.

Of course the essense of the example is that the range of Ω suddenly changed at one point i.e., the topology of the metric suddenly changed. This is perfectly compatible with the smoothness of Ω as it is only a weak symplectic form. This suggests a possible conjecture pointed out by Paul Chernoff: If Ω is such that the ranges of Ω_u are locally equivalent via an isomorphism, then Darboux's theorem should hold. This can be verified directly in case Ω comes from a metric which has locally equivalent ranges.

References

1. P. Chernoff and J. Marsden, *Hamiltonian systems and quantum mechanics* (in preparation).

2. J. M. Cook, *Complex Hilbertian structures on stable linear dynamical systems*, J. Math. Mech. **16** (1966), 339–349. MR **34** #625.

3. D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) **92** (1970), 102–163.

4. J. Marsden, *Hamiltonian one parameter groups*, Arch. Rational Mech. Anal. **28** (1967/68), 362–396. MR **37** #1735.

5. I. Segal, *Conjugacy to unitary groups within the infinite dimensional symplectic group*, Technical Report, Argonne National Laboratory, Argonne, Illinois, 1966.

6. A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*, Advances in Math. **6** (1971), 329–346.

7. ———, *Symplectic structures on Banach manifolds*, Bull. Amer. Math. Soc. **75** (1969), 1040–1041.

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