TWO NEW PROOFS OF LERCH’S FUNCTIONAL EQUATION

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Abstract.

One bright Sunday morning I went to church,
And there I met a man named Lerch.
We both did sing in jubilation,
For he did show me a new equation.

Two simple derivations of the functional equation of
$$\sum_{n=0}^{\infty} \exp\{2\pi inx\}(n+a)^{-s}$$
are given. The original proof is due to Lerch.

If \(x\) is real and \(0< a \leq 1\), define
$$\phi(x, a, s) = \sum_{n=0}^{\infty} \exp\{2\pi inx\}(n + a)^{-s},$$
where \(\sigma = \text{Re } s > 1\) if \(x\) is an integer, and \(\sigma > 0\) otherwise. Note that
\(\phi(0,a,s) = \zeta(s,a)\), the Hurwitz zeta-function. Furthermore, if \(a = 1\),
\(\phi(0,1,s) = \zeta(s)\), the Riemann zeta-function.

In 1887, Lerch [1] derived the following functional equation for \(\phi(x,a,s)\).

**Theorem.** Let \(0 < x < 1\). Then \(\phi(x,a,s)\) has an analytic continuation to the entire complex plane and is an entire function of \(s\). Furthermore, for all \(s\),
$$\phi(x, a, 1 - s) = (2\pi)^{-1}\Gamma(s)$$
(1)
$$\cdot \{\exp\left[\frac{1}{2}\pi is - 2\pi iax\right]\phi(-a, x, s) + \exp\left[-\frac{1}{2}\pi is + 2\pi ia(1-x)\right]\phi(a, 1-x, s)\}.$$

Lerch’s proof [1] of (1) depends upon the evaluation of a certain loop integral. Our objective here is to give two simple, new proofs of (1). The first proof uses contour integration; the second employs the Euler-Maclaurin summation formula. By slight variations of our methods, one can derive the corresponding result, namely Hurwitz’s formula, for \(\phi(0,a,s) = \zeta(s,a)\).
First Proof. Assume that \( s > 1 \) is real. With the aid of Euler's integral representation for \( \Gamma(s) \), it is easy to show that [1, pp. 19-20]

\[
\Gamma(s) \phi(x, a, s) = \int_0^\infty \frac{\exp[(1 - a)u - 2\pi ix]}{\exp[u - 2\pi ix] - 1} u^{s-1} \, du.
\]

If we put \( x = b + \frac{1}{2} \), then \(|b| < \frac{1}{2}\). Define

\[
F(z) = \frac{\pi \exp[2\pi ibz]}{(z + a)^s \sin(\pi z)},
\]

where the principal branch of \((z + a)^s\) is chosen. Choose \( c \) so that \(-a < c < 0\).

If \( m \) is a positive integer, let \( C_m \) denote the positively oriented contour consisting of the right half of the circle with center \((c, 0)\) and radius \( m + \frac{1}{2} - c \) together with the vertical diameter through \((c, 0)\). By the residue theorem,

\[
\frac{1}{2\pi i} \int_{C_m} F(z) \, dz = \sum_{n=0}^{m} \exp[2\pi inx](n + a)^{-s}.
\]

Let \( \Gamma_m \) denote the circular part of \( C_m \). Since \(|b| < \frac{1}{2}\), there is a constant \( M \), independent of \( m \), such that for \( z \) on \( \Gamma_m \), \( m \geq 1 \),

\[
\left| \frac{\exp[2\pi ibz]}{\sin(\pi z)} \right| \leq M.
\]

Hence,

\[
\left| \int_{\Gamma_m} F(z) \, dz \right| \leq \frac{\pi^2 (m + \frac{3}{2})M}{(m + \frac{1}{2})^s},
\]

which tends to 0 as \( m \) tends to \( \infty \) since \( s > 1 \). Thus, upon letting \( m \) tend to \( \infty \) in (3), we find that

\[
\phi(x, a, s) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) \, dz
\]

\[
= \int_c^{c+i\infty} \frac{\exp[2\pi ibz - \pi iz]}{(z + a)^s (\exp[-2\pi iz] - 1)} \, dz
\]

\[
+ \int_c^{c-i\infty} \frac{\exp[2\pi ibz + \pi iz]}{(z + a)^s (\exp[2\pi iz] - 1)} \, dz.
\]

We observe that the integrals in (4) converge uniformly in any compact set of the \( s \)-plane since \(|b| < \frac{1}{2}\). Hence, (4) shows that \( \phi(x, a, s) \) can be analytically continued to an entire function of \( s \).

Now suppose that \(-1 < s < 0\). We wish to let \( c \) approach \(-a\) in (4). In a neighborhood of \( z = -a \), we have

\[
\left| \frac{\exp[2\pi ibz \pm \pi iz]}{(z + a)^s (\exp[\pm 2\pi iz] - 1)} \right| \leq A |z + a|^{-s-1} \leq A |y|^{-s-1},
\]
where \( A \) is some positive number. Since \(-1 < s < 0\),
\[
\int_{0}^{\pi^1} |y|^{-s-1} \, dy < \infty.
\]
Hence, the two integrals on the right side of (4) converge uniformly on \(-a \leq c \leq -a+\epsilon\), for any number \( \epsilon > 0 \). We may then let \( c \) tend to \(-a \) in (4) to obtain
\[
\varphi(x, a, s) = i \exp\left[-\frac{1}{2} nis - 2\pi ab + \pi ia\right] \cdot \int_{0}^{\infty} \frac{\exp[-2\pi by + \pi y]}{y^{n}(\exp[2\pi y + 2\pi ia] - 1)} \, dy
\]
\[
- i \exp\left[\frac{1}{2} nis - 2\pi ab - \pi ia\right] \cdot \int_{0}^{\infty} \frac{\exp[2\pi by + \pi y]}{y^{n}(\exp[2\pi y - 2\pi ia] - 1)} \, dy.
\]
If we make the substitutions \( u=2\pi y \) and \( b=x-\frac{i}{2} \) and replace \( s \) by \( 1-s \), the above becomes, for \( s > 1 \),
\[
\varphi(x, a, 1-s) = \exp\left[\frac{1}{2} nis - 2\pi ia(x-1)\right](2\pi)^{-s}
\]
\[
\cdot \int_{0}^{\infty} \frac{\exp[-u(x-1)]u^{-n}}{\exp[u + 2\pi ia] - 1} \, du + \exp\left[-\frac{1}{2} nis - 2\pi iax\right](2\pi)^{-s}
\]
\[
\cdot \int_{0}^{\infty} \frac{\exp[u x]u^{-n}}{\exp[u - 2\pi ia] - 1} \, du.
\]
If we now use (2), (1) immediately follows for \( s > 1 \). By analytic continuation, (1) is valid for all \( s \).

Second Proof. Let \( f \) have a continuous first derivative on \([c, m]\), where \( m \) is a positive integer. Then we have the Euler-Maclaurin summation formula,
\[
\sum_{c < n \leq m} f(n) = \int_{c}^{m} f(u) \, du + \frac{1}{2} f(m) + (c - [c] - \frac{1}{2}) f(c)
\]
\[
+ \int_{c}^{m} \left(u - [u] - \frac{1}{2}\right) f'(u) \, du.
\]
Put \( c=1-a \) and \( f(u)=(u+a)^{-s} \exp(2\pi iux) \), where \( s > 0 \). Then, upon letting \( m \) tend to \( \infty \) in (5), we obtain
\[
\varphi(x, a, s) = \int_{1-a}^{\infty} (u + a)^{-s} \exp[2\pi iux] \, du
\]
\[
+ (\frac{1}{2} - a) \exp[2\pi i(1-a)]
\]
\[
- s \int_{1-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^{s+1}} \exp[2\pi iux] \, du
\]
\[
+ 2\pi i \int_{1-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^{s}} \exp[2\pi iux] \, du.
\]
The three integrals on the right side of (6) all converge for \( \sigma > 0 \) by Dirichlet's test.

First, assume that \( 0 < \sigma < 1 \). Since

\[
[u] - u + \frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin(2\pi nu)}{\pi n}
\]

if \( u \) is not an integer, we have formally

\[
\int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^s} \exp[2\pi iux] \, du
\]

\[
= -\frac{1}{\pi} \int_{0}^{\infty} u^{-s} \exp[2\pi i(u - a)x] \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n(u - a)) \, du
\]

\[
= \frac{\exp[-2\pi iax]}{2\pi i} \sum_{n=1}^{\infty} \frac{1}{n} \left( \exp[2\pi i\alpha] \int_{0}^{\infty} u^{-s} \exp[2\pi iu(x - n)] \, du - \exp[-2\pi i\alpha] \int_{0}^{\infty} u^{-s} \exp[2\pi iu(x + n)] \, du \right).
\]

We must justify the inversion in order of summation and integration. Since the Fourier series in (7) is boundedly convergent, the inversion is justified if we integrate over \((0, b)\), where \( b \) is any finite number [2, p. 41]. We need then only show that, for \( 0 < \alpha < 1 \),

\[
\lim_{b \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \left( \exp[2\pi i\alpha] \int_{0}^{\infty} u^{-s} \exp[2\pi iu(x - n)] \, du - \exp[-2\pi i\alpha] \int_{0}^{\infty} u^{-s} \exp[2\pi iu(x + n)] \, du \right) = 0.
\]

Upon an integration by parts,

\[
\int_{0}^{\infty} u^{-s} \exp[2\pi iu(x - n)] \, du = O(b^{-\sigma}/n)
\]

\[
+ \frac{s}{2\pi i(x - n)} \int_{0}^{\infty} u^{-s-1} \exp[2\pi iu(x - n)] \, du = O(b^{-\sigma}/n),
\]

as \( b \) tends to \( \infty \). By the same argument we obtain the same \( O \)-estimate for the integrals involving \( \exp[2\pi iu(x + n)] \). Hence, (9) now easily follows.

Now, if \( 0 < \sigma < 1 \) and \( d \neq 0 \) is real, we have [2, pp. 107–103]

\[
\int_{0}^{\infty} u^{-s} \exp[i\alpha u] \, du = \Gamma(1 - s) |d|^{-s-1} \exp[\frac{1}{2} \pi i(1 - s) \text{sgn } d].
\]
Thus, (8) becomes

$$\int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^s} \exp[2\pi iux] \, du$$

$$= - (2\pi)^{-s} \Gamma(1 - s) \exp\left[\frac{1}{2} \pi is - 2\pi iax\right]$$

$$\cdot \sum_{n=1}^{\infty} \left( \frac{\exp[2\pi i a] \exp[-\pi is - 2\pi i a]}{n(n-x)^{1-s} + n(n+x)^{1-s}} \right).$$

Using (10) again, we have, for $0 < \sigma < 1$,

$$\int_{-a}^{\infty} (u + a)^{-s} \exp[2\pi iux] \, du$$

$$= \exp[-2\pi iax] \int_{-a}^{\infty} u^{-s} \exp[2\pi iux] \, du$$

$$= \Gamma(1 - s)(2\pi x)^{-s} \exp[\pi i(s - 1) - 2\pi i ax].$$

Hence, substituting (11) and (12) into (6), we obtain, for $0 < \sigma < 1$,

$$\varphi(x, a, x) = \Gamma(1 - s)(2\pi x)^{-s} \exp[\frac{1}{2} \pi i(1 - s) - 2\pi i ax]$$

$$- \int_{-a}^{1-a} (u + a)^{-s} \exp[2\pi iux] \{1 + 2\pi i(x - [x] - \frac{1}{2}) \} \, du$$

$$+ (\frac{1}{a} - a) \exp[2\pi i x(1 - a)] - s \int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^{s+1}} \exp[2\pi iux] \, du$$

$$+ x(2\pi)^{-s} \Gamma(1 - s) \exp[\frac{1}{2} \pi i(s - 1) - 2\pi i ax]$$

$$\cdot \sum_{n=1}^{\infty} \left( \frac{\exp[2\pi i a] \exp[-\pi is - 2\pi i a]}{n(n-x)^{1-s} + n(n+x)^{1-s}} \right).$$

We next observe that the infinite series on the right side of (13) converges absolutely and uniformly on any compact subset of the strip $-1 < \sigma < 1$. By Dirichlet’s test, the integrals on the right side of (13) converge uniformly on any compact subset of the strip $-1 < \sigma < 1$. Hence, by analytic continuation, (13) is valid for $-1 < \sigma < 1$. Assume now that $-1 < \sigma < 0$. Replacing $s$ by $s+1$ in (11), we have

$$\int_{-a}^{\infty} \frac{u - [u] - \frac{1}{2}}{(u + a)^{s+1}} \exp[2\pi iux] \, du$$

$$= (2\pi)^{-s} \Gamma(-s) \exp[\frac{1}{2} \pi i(s - 1) - 2\pi i ax]$$

$$\cdot \sum_{n=1}^{\infty} \left( \frac{\exp[2\pi i a] \exp[-i\pi s - 2\pi i a]}{n(n-x)^{s} - n(n+x)^{s}} \right).$$
Substitute (14) into (13), use the functional equation of \( \Gamma(s) \), and observe that \( x/[n(n-x)] + 1/n = 1/(n-x) \) and \( x/[n(n+x)] - 1/n = -1/(n+x) \). For \(-1 < \sigma < 0\) we arrive at

\[
\varphi(x, a, s) = \frac{1}{a^s} \Gamma(1-x)(2\pi x)^{s-1} \exp[\frac{1}{2} \pi i (1-s) - 2\pi i a x] \\
- \int_{-a}^{1-a} (u + a)^{-s} \exp[2\pi i u x] \\
\cdot \left\{ \left[ 1 + 2\pi i x (u - [u] - \frac{1}{2}) - s (u + a)^{-1} (u - [u] - \frac{1}{2}) \right] du \\
+ (\frac{1}{2} - a) \exp[2\pi i x (1-a)] \\
+ (2\pi)^{-1} \Gamma(1-s) \exp[\frac{1}{2} \pi i (s - 1) - 2\pi i a x] \\
\cdot \left\{ \sum_{n=1}^{\infty} \frac{\exp[2\pi i n a]}{n (n-x)^{1-s}} - \sum_{n=1}^{\infty} \frac{\exp[-\pi i s - 2\pi i n a]}{(n+x)^{1-s}} \right\} \right\}.
\]

Observe that the first expression on the right side of (15) corresponds to the term \( n=0 \) for the second series on the right side of (15). In the first series replace \( n \) by \( n+1 \). By an elementary calculation the second expression on the right side of (15) is seen to be

\[-a^{-s} - (\frac{1}{2} - a) \exp[2\pi i x (1-a)].\]

Upon these simplifications, (15) now becomes, for \(-1 < \sigma < 0\),

\[
\varphi(x, a, s) = (2\pi)^{-1} \Gamma(1-s) \exp[\frac{1}{2} \pi i (s - 1) - 2\pi i a x] \\
\cdot \left\{ \exp[2\pi i a] \varphi(a, 1-x, 1-s) - \exp[-\pi i s] \varphi(-a, x, 1-s) \right\}.
\]

By analytic continuation (16) is valid for all \( s \). Now replace \( s \) by \( 1-s \) in (16) to obtain (1).

References

1. M. Lerch, Note sur la fonction \( \Phi(w, x, s) = \sum_{k=0}^{\infty} e^{\pi i k w}/(w+k)^s \), Acta Math. 11 (1887), 19–24.