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TWO NEW PROOFS OF LERCH'S FUNCTIONAL EQUATION

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ABSTRACT.

One bright Sunday morning I went to church,
And there I met a man named Lerch.
We both did sing in jubilation,
For he did show me a new equation.

Two simple derivations of the functional equation of

$$\sum_{n=0}^{\infty} \exp[2\pi inx](n+a)^{-s}$$

are given. The original proof is due to Lerch.

If x is real and $0<a\leqq 1$, define

$$\varphi(x, a, s) = \sum_{n=0}^{\infty} \exp[2\pi inx](n + a)^{-s},$$

where $\sigma = \operatorname{Re} s > 1$ if x is an integer, and $\sigma > 0$ otherwise. Note that $\varphi(0, a, s) = \zeta(s, a)$, the Hurwitz zeta-function. Furthermore, if $a=1$, $\varphi(0, 1, s) = \zeta(s)$, the Riemann zeta-function.

In 1887, Lerch [**1**] derived the following functional equation for $\varphi(x, a, s)$.

THEOREM. *Let $0<x<1$. Then $\varphi(x, a, s)$ has an analytic continuation to the entire complex plane and is an entire function of s. Furthermore, for all s,*

$$\begin{aligned} \varphi(x, a, 1 - s) = (2\pi)^{-s}\Gamma(s) \\ \cdot \{\exp[\tfrac{1}{2}\pi is - 2\pi iax]\varphi(-a, x, s) \\ + \exp[-\tfrac{1}{2}\pi is + 2\pi ia(1 - x)]\varphi(a, 1 - x, s)\}. \end{aligned} \tag{1}$$

Lerch's proof [**1**] of (1) depends upon the evaluation of a certain loop integral. Our objective here is to give two simple, new proofs of (1). The first proof uses contour integration; the second employs the Euler-Maclaurin summation formula. By slight variations of our methods, one can derive the corresponding result, namely Hurwitz's formula, for $\varphi(0, a, s) = \zeta(s, a)$.

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FIRST PROOF. Assume that $s>1$ is real. With the aid of Euler's integral representation for $\Gamma(s)$, it is easy to show that [**1**, pp. 19–20]

$$\Gamma(s)\varphi(x, a, s) = \int_0^\infty \frac{\exp[(1-a)u - 2\pi i x]}{\exp[u - 2\pi i x] - 1} u^{s-1}\, du. \tag{2}$$

If we put $x=b+\frac{1}{2}$, then $|b|<\frac{1}{2}$. Define

$$F(z) = \frac{\pi \exp[2\pi i b z]}{(z+a)^s \sin(\pi z)},$$

where the principal branch of $(z+a)^s$ is chosen. Choose c so that $-a<c<0$. If m is a positive integer, let C_m denote the positively oriented contour consisting of the right half of the circle with center $(c, 0)$ and radius $m+\frac{1}{2}-c$ together with the vertical diameter through $(c, 0)$. By the residue theorem,

$$\frac{1}{2\pi i}\int_{C_m} F(z)\, dz = \sum_{n=0}^{m} \exp[2\pi i n x](n+a)^{-s}. \tag{3}$$

Let Γ_m denote the circular part of C_m. Since $|b|<\frac{1}{2}$, there is a constant M, independent of m, such that for z on Γ_m, $m\geqq 1$,

$$\left|\frac{\exp[2\pi i b z]}{\sin(\pi z)}\right| \leqq M.$$

Hence,

$$\left|\int_{\Gamma_m} F(z)\, dz\right| \leqq \frac{\pi^2(m+\frac{3}{2})M}{(m+\frac{1}{2})^s},$$

which tends to 0 as m tends to ∞ since $s>1$. Thus, upon letting m tend to ∞ in (3), we find that

$$\begin{aligned} \varphi(x, a, s) &= -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)\, dz \\ &= \int_c^{c+i\infty} \frac{\exp[2\pi i b z - \pi i z]}{(z+a)^s(\exp[-2\pi i z] - 1)}\, dz \\ &\quad + \int_c^{c-i\infty} \frac{\exp[2\pi i b z + \pi i z]}{(z+a)^s(\exp[2\pi i z] - 1)}\, dz. \end{aligned} \tag{4}$$

We observe that the integrals in (4) converge uniformly in any compact set of the s-plane since $|b|<\frac{1}{2}$. Hence, (4) shows that $\varphi(x, a, s)$ can be analytically continued to an entire function of s.

Now suppose that $-1<s<0$. We wish to let c approach $-a$ in (4). In a neighborhood of $z=-a$, we have

$$\left|\frac{\exp[2\pi i b z \pm \pi i z]}{(z+a)^s(\exp[\pm 2\pi i z] - 1)}\right| \leqq A\,|z+a|^{-s-1} \leqq A\,|y|^{-s-1},$$

where A is some positive number. Since $-1<s<0$,

$$\int_0^{\mp 1} |y|^{-s-1}\, dy < \infty.$$

Hence, the two integrals on the right side of (4) converge uniformly on $-a\leqq c\leqq -a+\varepsilon$, for any number $\varepsilon>0$. We may then let c tend to $-a$ in (4) to obtain

$$\begin{aligned}\varphi(x, a, s) = {} & i \exp[-\tfrac{1}{2}\pi is - 2\pi iab + \pi ia] \\ & \cdot \int_0^\infty \frac{\exp[-2\pi by + \pi y]}{y^s(\exp[2\pi y + 2\pi ia] - 1)}\, dy \\ & - i \exp[\tfrac{1}{2}\pi is - 2\pi iab - \pi ia] \\ & \cdot \int_0^\infty \frac{\exp[2\pi by + \pi y]}{y^s(\exp[2\pi y - 2\pi ia] - 1)}\, dy.\end{aligned}$$

If we make the substitutions $u=2\pi y$ and $b=x-\frac{1}{2}$ and replace s by $1-s$, the above becomes, for $s>1$,

$$\begin{aligned}\varphi(x, a, 1-s) = {} & \exp[\tfrac{1}{2}\pi is - 2\pi ia(x-1)](2\pi)^{-s} \\ & \cdot \int_0^\infty \frac{\exp[-u(x-1)]u^{s-1}}{\exp[u + 2\pi ia] - 1}\, du \\ & + \exp[-\tfrac{1}{2}\pi is - 2\pi iax](2\pi)^{-s} \\ & \cdot \int_0^\infty \frac{\exp[ux]u^{s-1}}{\exp[u - 2\pi ia] - 1}\, du.\end{aligned}$$

If we now use (2), (1) immediately follows for $s>1$. By analytic continuation, (1) is valid for all s.

Second Proof. Let f have a continuous first derivative on $[c, m]$, where m is a positive integer. Then we have the Euler-Maclaurin summation formula,

$$\tag{5} \begin{aligned}\sum_{c<n\leqq m} f(n) = {} & \int_c^m f(u)\, du + \tfrac{1}{2}f(m) + (c - [c] - \tfrac{1}{2})f(c) \\ & + \int_c^m (u - [u] - \tfrac{1}{2})f'(u)\, du.\end{aligned}$$

Put $c=1-a$ and $f(u)=(u+a)^{-s}\exp(2\pi iux)$, where $\sigma>0$. Then, upon letting m tend to ∞ in (5), we obtain

$$\tag{6} \begin{aligned}\varphi(x, a, s) - a^{-s} = {} & \int_{1-a}^\infty (u + a)^{-s} \exp[2\pi iux]\, du \\ & + (\tfrac{1}{2} - a)\exp[2\pi ix(1-a)] \\ & - s\int_{1-a}^\infty \frac{u - [u] - \frac{1}{2}}{(u + a)^{s+1}} \exp[2\pi iux]\, du \\ & + 2\pi ix \int_{1-a}^\infty \frac{u - [u] - \frac{1}{2}}{(u + a)^s} \exp[2\pi iux]\, du.\end{aligned}$$

The three integrals on the right side of (6) all converge for $\sigma>0$ by Dirichlet's test.

First, assume that $0<\sigma<1$. Since

$$[u]-u+\tfrac{1}{2}=\sum_{n=1}^{\infty}\frac{\sin(2\pi nu)}{\pi n} \tag{7}$$

if u is not an integer, we have formally

$$\begin{aligned}\int_{-a}^{\infty}&\frac{u-[u]-\frac{1}{2}}{(u+a)^s}\exp[2\pi iux]\,du\\ &=-\frac{1}{\pi}\int_0^{\infty}u^{-s}\exp[2\pi i(u-a)x]\sum_{n=1}^{\infty}\frac{1}{n}\sin(2\pi n\{u-a\})\,du\\ &=\frac{\exp[-2\pi iax]}{2\pi i}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\exp[2\pi ina]\int_0^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du\right.\\ &\qquad\left.-\exp[-2\pi ina]\int_0^{\infty}u^{-s}\exp[2\pi iu(x+n)]\,du\right\}.\end{aligned} \tag{8}$$

We must justify the inversion in order of summation and integration. Since the Fourier series in (7) is boundedly convergent, the inversion is justified if we integrate over $(0, b)$, where b is any finite number [**2**, p. 41]. We need then only show that, for $0<\sigma<1$,

$$\begin{aligned}\lim_{b\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}&\left\{\exp[2\pi ina]\int_b^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du\right.\\ &\left.-\exp[-2\pi ina]\int_b^{\infty}u^{-s}\exp[2\pi iu(x+n)]\,du\right\}=0.\end{aligned} \tag{9}$$

Upon an integration by parts,

$$\begin{aligned}\int_b^{\infty}u^{-s}\exp[2\pi iu(x-n)]\,du&=O(b^{-\sigma}/n)\\ &+\frac{s}{2\pi i(x-n)}\int_b^{\infty}u^{-s-1}\exp[2\pi iu(x-n)]\,du=O(b^{-\sigma}/n),\end{aligned}$$

as b tends to ∞. By the same argument we obtain the same O-estimate for the integrals involving $\exp\{2\pi iu(x+n)\}$. Hence, (9) now easily follows.

Now, if $0<\sigma<1$ and $d\neq0$ is real, we have [**2**, pp. 107–108]

$$\int_0^{\infty}u^{-s}\exp[idu]\,du=\Gamma(1-s)\,|d|^{s-1}\exp[\tfrac{1}{2}\pi i(1-s)\operatorname{sgn}d]. \tag{10}$$

Thus, (8) becomes

$$\int_{-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^s} \exp[2\pi iux]\,du$$
$$(11) \qquad = -(2\pi)^{s-2}\Gamma(1-s)\exp[\tfrac{1}{2}\pi is - 2\pi iax] \cdot \sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{1-s}} + \frac{\exp[-\pi is - 2\pi ina]}{n(n+x)^{1-s}}\right\}.$$

Using (10) again, we have, for $0<\sigma<1$,

$$\int_{-a}^{\infty} (u+a)^{-s} \exp[2\pi iux]\,du$$
$$(12) \qquad = \exp[-2\pi iax]\int_0^{\infty} u^{-s}\exp[2\pi iux]\,du$$
$$= \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax].$$

Hence, substituting (11) and (12) into (6), we obtain, for $0<\sigma<1$,

$$\varphi(x,a,x) - a^{-s}$$
$$= \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax]$$
$$-\int_{-a}^{1-a}(u+a)^{-s}\exp[2\pi iux]\{1 + 2\pi ix(u-[u]-\tfrac{1}{2})\}\,du$$
$$(13) \qquad + (\tfrac{1}{2}-a)\exp[2\pi ix(1-a)] - s\int_{1-a}^{\infty}\frac{u-[u]-\frac{1}{2}}{(u+a)^{s+1}}\exp[2\pi iux]\,du$$
$$+ x(2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax]$$
$$\cdot\sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{1-s}} + \frac{\exp[-\pi is - 2\pi ina]}{n(n+x)^{1-s}}\right\}.$$

We next observe that the infinite series on the right side of (13) converges absolutely and uniformly on any compact subset of the strip $-1<\sigma<1$. By Dirichlet's test, the integrals on the right side of (13) converge uniformly on any compact subset of the strip $-1<\sigma<1$. Hence, by analytic continuation, (13) is valid for $-1<\sigma<1$. Assume now that $-1<\sigma<0$. Replacing s by $s+1$ in (11), we have

$$\int_{-a}^{\infty} \frac{u-[u]-\frac{1}{2}}{(u+a)^{s+1}} \exp[2\pi iux]\,du$$
$$(14) \qquad = (2\pi)^{s-1}\Gamma(-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \cdot \sum_{n=1}^{\infty}\left\{\frac{\exp[2\pi ina]}{n(n-x)^{-s}} - \frac{\exp[-i\pi s - 2\pi ina]}{n(n+x)^{-s}}\right\}.$$

Substitute (14) into (13), use the functional equation of $\Gamma(s)$, and observe that $x/\{n(n-x)\}+1/n=1/(n-x)$ and $x/\{n(n+x)\}-1/n=-1/(n+x)$. For $-1<\sigma<0$ we arrive at

$$\begin{aligned}
&\varphi(x, a, s) - a^{-s} \\
&\quad = \Gamma(1-s)(2\pi x)^{s-1}\exp[\tfrac{1}{2}\pi i(1-s) - 2\pi iax] \\
&\qquad - \int_{-a}^{1-a} (u+a)^{-s}\exp[2\pi iux] \\
&\qquad \cdot \{1 + 2\pi ix(u-[u]-\tfrac{1}{2}) - s(u+a)^{-1}(u-[u]-\tfrac{1}{2})\}\,du \qquad (15)\\
&\qquad + (\tfrac{1}{2}-a)\exp[2\pi ix(1-a)] \\
&\qquad + (2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \\
&\qquad \cdot \left\{\sum_{n=1}^{\infty}\frac{\exp[2\pi ina]}{(n-x)^{1-s}} - \sum_{n=1}^{\infty}\frac{\exp[-\pi is - 2\pi ina]}{(n+x)^{1-s}}\right\}.
\end{aligned}$$

Observe that the first expression on the right side of (15) corresponds to the term $n=0$ for the second series on the right side of (15). In the first series replace n by $n+1$. By an elementary calculation the second expression on the right side of (15) is seen to be

$$-a^{-s} - (\tfrac{1}{2}-a)\exp[2\pi ix(1-a)].$$

Upon these simplifications, (15) now becomes, for $-1<\sigma<0$,

$$\begin{aligned}
&\varphi(x, a, x) \\
&\quad = (2\pi)^{s-1}\Gamma(1-s)\exp[\tfrac{1}{2}\pi i(s-1) - 2\pi iax] \qquad (16)\\
&\qquad \cdot \{\exp[2\pi ia]\varphi(a, 1-x, 1-s) - \exp[-\pi is]\varphi(-a, x, 1-s)\}.
\end{aligned}$$

By analytic continuation (16) is valid for all s. Now replace s by $1-s$ in (16) to obtain (1).

References

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