A NOTE ON THE UNIQUENESS OF RINGS OF COEFFICIENTS IN POLYNOMIAL RINGS

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Abstract. We say that the ring $A$ is of transcendence degree $n$ over its subfield $k$ if for every prime $P \subset A$ the transcendence degree of $A/P$ over $k$ is at most $n$ and equality is attained for some $P$. In this paper we prove the following: Suppose $A$ is a noetherian ring of transcendence degree one over its subfield $k$. Then if $B$ is any ring such that the polynomial rings

$$A[X_1, \ldots, X_n] \text{ and } B[Y_1, \ldots, Y_n]$$

are isomorphic, $A$ is isomorphic to $B$. Moreover if $A$ has no non-trivial idempotents then either $A$ is isomorphic to the polynomials in one variable over a local artinian ring or, modulo the nil radical, the given isomorphism takes $A$ onto $B$.

This is a continuation of the investigation, begun in [CE] and [AEH], of the extent to which a ring is determined by its polynomial rings. More precisely, we study the following: Suppose $A$ is a ring and $R = A[X_1, \ldots, X_n]$ is a ring of polynomials over $A$. If $B$ is a ring and $S = B[Y_1, \ldots, Y_n]$ is a ring of polynomials over $B$ such that $R$ is isomorphic to $S$, does it follow that $A$ is isomorphic to $B$? In [CE] and [AEH] it is shown that for a fairly extensive class of rings $A$, the answer is affirmative. Our main purpose here is to extend this class of rings to include rings of Krull dimension one which are finitely generated over a field. This has been done for the case of rings without zero divisors in [AEH].

Unless there is a statement to the contrary all rings are assumed to be commutative and to possess an identity $1 \neq 0$. If $R$ is a ring, we write $A[X_1, \ldots, X_n] = A^{(n)}$ when we want it to be understood or emphasized that the $X_i$'s are algebraically independent over $A$. If $A$ is a ring, we let $N(A)$ denote the nil radical of $A$ and $A^*$ the reduced ring $A/N(A)$.

Let $A$ be a ring and $R = A[X_1, \ldots, X_n] = A^{(n)}$. We have

$$N(R) = N(A)(R[X_1, \ldots, X_n]) \text{ and } R^* = A^*[\bar{X}_1, \ldots, \bar{X}_n] = A^{*(n)}.$$

If $A$ and $B$ are rings and there is an isomorphism

$$A^{(n)} = A[X_1, \ldots, X_n] \xrightarrow{\sigma} B[Y_1, \ldots, Y_n] = B^{(n)},$$

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then since $\sigma$ maps the nil radical of $A^{(n)}$ onto the nil radical of $B^{(n)}$ we have an induced isomorphism $\sigma^*$ and a commutative diagram with exact rows and columns

\[
\begin{array}{ccc}
0 & \to & N(A)[X_1, \ldots, X_n] \\
\sigma & \to & N(B)[Y_1, \ldots, Y_n] \\
0 & \to & A[X_1, \ldots, X_n] \\
\sigma & \to & B[Y_1, \ldots, Y_n] \\
0 & \to & A^*[X_1, \ldots, X_n] \\
\sigma^* & \to & B^*[Y_1, \ldots, Y_n] \\
0 & \to & 0 \\
\end{array}
\]

With this notation we give the following:

1. **Definition.** A ring $A$ is said to be **strongly invariant** if for any ring $B$ and any isomorphism of polynomial rings $\sigma : A[X_1, \ldots, X_n] \to B[Y_1, \ldots, Y_n]$ we have $\sigma(A^*) = B^*$.

A weaker concept is:

2. **Definition.** A ring $A$ is said to be **invariant** if for any ring $B$ such that there is an isomorphism of the polynomial rings $A[X_1, \ldots, X_n]$ and $B[Y_1, \ldots, Y_n]$, we have $B$ isomorphic to $A$.

Definition (1) is consistent with those of [CE] and [AEH]. In the case where $A$ has no nilpotent elements, and in particular where $A$ is a domain, (1) simply asserts that the isomorphism $\sigma$ takes $A$ onto $B$. With these definitions it is not obvious that strongly invariant rings are invariant. Before proving this we make the following useful observation.

3. If $A$ is a ring, then $A$ is invariant (resp. strongly invariant) if and only if whenever $B$ is a ring such that $R = A^{(n)} = A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n] = B^{(n)}$ then $A$ is isomorphic to $B$ (resp. $A^* = B^*$ in $R^*$).

The proof is straightforward and we omit it.

4. Strongly invariant rings are invariant.

**Proof.** Suppose $A$ is a strongly invariant ring. Let $R = A[X_1, \ldots, X_n] = A^{(n)} = B[Y_1, \ldots, Y_n] = B^{(n)}$ for some ring $B$, then $A^* = B^*$ by (3). We claim this implies that $A[X_1, \ldots, X_n] = A[Y_1, \ldots, Y_n]$. Since $A^{(n)} = A^*[X_1, \ldots, X_n] = A^*[Y_1, \ldots, Y_n]$, we have, for each $i$, $X_i = g_i(Y_1, \ldots, Y_n) + n_i(X_1, \ldots, X_n)$, so

\[
X_i = g_i(Y_1, \ldots, Y_n) + n_i(X_1, \ldots, X_n)
\]

where $n_i$ has nilpotent coefficients which are elements of $A$. We also have, for each $i$,

\[
Y_i = h_i(X_1, \ldots, X_n).
\]
Let Δ be the set of all the coefficients of the \( g_i, n_j \) and \( h_k \). Set \( A_0 = \Pi[\Delta] \) where \( \Pi \) is the prime subring of \( A \). Then

\[
A_0[Y_1, \ldots, Y_n] \subseteq A_0[X_1, \ldots, X_n] = A_0(n)
\]

and

\[
A_0^*[\overline{Y}_1, \ldots, \overline{Y}_n] = A_0^*[\overline{X}_1, \ldots, \overline{X}_n].
\]

Now \( A_0[Y_1, \ldots, Y_n] = A_0[X_1, \ldots, X_n] \). For

\[
A_0[X_1, \ldots, X_n] \subseteq A_0[Y_1, \ldots, Y_n] + N(A_0)[Y_1, \ldots, Y_n],
\]

and iterating this we have

\[
A_0[X_1, \ldots, X_n] \subseteq \sum_{j=0}^{k-1} [N(A_0)]^j[Y_1, \ldots, Y_n].
\]

Since \( A_0 \) is noetherian, there is a \( k \) such that \([N(A_0)]^k = 0\), and therefore \( A[X_1, \ldots, X_n] = A[Y_1, \ldots, Y_n] \). Now we claim that the \( Y_i \) are algebraically independent over \( A \). If \( c(Y_1, \ldots, Y_n) = 0 \) is an equation of algebraic dependence and \( A_1 \) is the subring of \( A \) generated by \( A_0 \) and the coefficients of \( c \), we have

\[
A_1[X_1, \ldots, X_n] = A_1[Y_1, \ldots, Y_n].
\]

Since the \( X_i \) are algebraically independent over \( A_1 \) there is an \( A_1 \)-homomorphism

\[
A_1[X_1, \ldots, X_n] \xrightarrow{\tau} A_1[Y_1, \ldots, Y_n]
\]

which takes \( X_i \) to \( Y_i \). Then \( c(X_1, \ldots, X_n) \) must be in the kernel of \( \tau \). Since \( A_1[Y_1, \ldots, Y_n] = A_1[X_1, \ldots, X_n] \), \( \tau \) is a homomorphism of \( A_1[X_1, \ldots, X_n] \) onto itself. Since any homomorphism of a noetherian ring onto itself is an isomorphism, \( \tau \) is an isomorphism and hence all of the coefficients of \( c \) are zero. Thus the \( Y_i \)'s are algebraically independent over \( A \). Now we have immediately that \( A \) is isomorphic to \( B \), for if \( \mathfrak{M} \) is the ideal of \( R \) generated by \( \{ Y_1, \ldots, Y_n \} \), then \( R/\mathfrak{M} \) is isomorphic to both \( A \) and \( B \). This concludes the proof of (3).

In \([AEH]\) the following is proved:

If \( A \) is a domain which is of transcendence degree one over a field, then \( A \) is invariant. \( A \) is either strongly invariant or there is a field \( k \) such that \( A = k[t] = k(t) \).

In the sequel any reference to \([AEH]\) is an appeal to this result. This shows that if \( t \) is transcendental over the field \( k \), then \( k[t] \) is an invariant ring which is not strongly invariant, for \( k[t][X] = k[X][t] \) but \( k[t] \neq k[X] \).

(5) **Remark.** We do not know of an example of a commutative ring with identity which is not invariant. However there are simple examples
of noninvariant rings without identity. In fact let $G$ be any abelian group such that $G$ is not isomorphic to $G \oplus G$. Make each of these groups into a ring defining the product of any two elements to be zero. Then $G[X]$ and $(G \oplus G)[Y]$ are each isomorphic to a direct sum of countably many copies of $G$, hence they are isomorphic.

(6) If $A$ is a noetherian ring then $A$ is invariant if and only if whenever $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n] = B(n)$, there is a homomorphism from $A$ onto $B$.

Proof. Such a homomorphism can be extended to a mapping of $A[X_1, \ldots, X_n]$ onto $B[Y_1, \ldots, Y_n]$ which would be a homomorphism of $A[X_1, \ldots, X_n]$ onto itself. Since any onto endomorphism of a noetherian ring is an isomorphism, the original homomorphism must have been an isomorphism.

(7) Any finite direct sum of invariant rings is invariant; any finite direct sum of strongly invariant rings is strongly invariant.

Proof. Let $A = A_1 \oplus \cdots \oplus A_k$ and $A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n] = B(n)$, and let $e_i$ be the identity of $A_i$. Since $e_i$ is idempotent, it is easy to see that $e_i B e_i = B e_i e_i$ for each $i$. Thus $B e_1 + B e_2 + \cdots + B e_k$ is a direct sum decomposition of $B$ and $A e_i [e_i X_1, \ldots, e_i X_n] = B e_i [e_i Y_1, \ldots, e_i Y_n] = e_i R$ becomes

$$A^{(n)} = A_i [\bar{X}_1, \ldots, \bar{X}_n] = B_i [\bar{Y}_1, \ldots, \bar{Y}_n] = B_i^{(n)}.$$ 

If each $A_i$ is strongly invariant then $A_i$ is isomorphic to $B_i$ for each $i$ and consequently $A$ is isomorphic to $B$.

We say that the ring $R$ is of transcendence degree $n$ over a field $F$ if $R$ contains $F$, deg tr $R/p \leq n$ for each prime ideal $p \subseteq R$, and there is a prime ideal $p$ for which equality is attained.

(8) Let $A$ be a one dimensional noetherian ring which is of transcendence degree one over a field and suppose $A$ has no nontrivial idempotents. Then if $A$ has at least two primes of height zero, $A$ is strongly invariant.

Proof. Let $A^{(n)} = A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n] = B(n)$. We wish to show $A^* = B^*$. Let $q_1, \ldots, q_k$ be the height zero primes of $A$. Then since $\mathfrak{A}_i = q_i [X_1, \ldots, X_n]$ is also of height zero, there are primes $p_1, \ldots, p_k$ of $B$ such that $p_i [Y_1, \ldots, Y_n] = q_i [X_1, \ldots, X_n]$ and the $p_i$ are precisely the height zero primes of $B$. Thus we have a diagram:

$$A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n] = R,$$

$$A/q_i [\bar{X}_1, \ldots, \bar{X}_n] = B/p_i [\bar{Y}_1, \ldots, \bar{Y}_n] = R/\mathfrak{A}_i$$

and, to see that $A^* = B^*$, it is sufficient to see that $A/q_i = B/p_i$ in $R/\mathfrak{A}_i$ for
each i. Since A has no nontrivial idempotents, A cannot be represented as a direct sum and thus $q_i$ and $b = \prod_{i=2}^{k} q_i$ are contained in a common maximal ideal. For if not we would have $A = A/q_i \oplus A/b$ whenever $(N(A))^i = 0$. Let M be a maximal ideal of A such that $q_i + b \subseteq M$. Since M is prime, at least one of the $q_i$ ($i \neq 1$) is contained in M, say $q_2 \subseteq M$. Now $q_1 + q_2 \subseteq M$ implies $q_1[X_1, \ldots, X_n]$, $q_2[X_1, \ldots, X_n] \subseteq M[X_1, \ldots, X_n]$, and therefore $p_1 + p_2 \subseteq M[X_1, \ldots, X_n] \cap \mathcal{B} = N$. Since $p_1$ and $p_2$ are distinct, $p_1 + p_2$ is not of height zero and therefore N is a maximal ideal of B such that $N[Y_1, \ldots, Y_n] = M[X_1, \ldots, X_n]$. Now in

$$R_1 = A/q_1[\bar{X}_1, \ldots, \bar{X}_n] = (A/q_1)^{(n)} = B/p_1[\bar{Y}_1, \ldots, \bar{Y}_n] = (B/p_1)^{(n)}$$

we have that either $A/q_1 = B/p_1$ or each is of the form $k[T] = k^{(1)}$ for some field k [AEH]. But even in the latter case, they are equal. For each is a P.I.D., so set $M/q_1 = (a)$ and $N/p_1 = (b)$. Then $aR_1 = bR_1$ and so a and b differ only by a unit of R. Since a generates a nonzero prime of $A/q_1$, it is transcendental over k. Thus it is a transcendence base for both $A/q_1$ and $B/p_1$ over k. But each of $A/q_1$ and $B/p_1$ is the algebraic closure of $k[a]$ in $R_1$, and hence $A/q_1 = B/p_1$. Since we could have chosen any prime of height zero to be $q_1$, it follows that $A^* = B^*$ and we have completed the argument.

(9) Remark. In case $A$ is a reduced affine ring over a field, the previous argument has a geometric interpretation. One views $A$ as the coordinate algebra of an affine curve $\Gamma_A$. Then $A[X]$ is the coordinate algebra of the cylinder $C_A$ over $\Gamma_A$ (see Figure 1).

![Figure 1](http://example.com/image.png)
If $\Gamma_A$ is not irreducible, then it is the union of a finite number of irreducible components $\Gamma_1, \cdots, \Gamma_k$ where the $\Gamma_i$'s correspond to the height zero primes $q_i$. The assumption that $A$ has no nontrivial idempotents is equivalent to the fact that our curve $\Gamma_A$ is connected in the Zariski topology. In the proof of (8) the maximal ideal $M$ such $q_1 + q_2 \subseteq M$ corresponds to a point $m$ on the intersection of the components $\Gamma_1$ and $\Gamma_2$. The assumption $A[X] \cong B[Y]$ corresponds to the existence of a birational mapping $\phi$ from $C_A$ onto a cylinder $C_B$ over some curve $\Gamma_B$. With reference to Figure 1 the proof essentially shows that lines such as $L$ on $C_A$ (L corresponds to $M[X]$) must be taken to similar lines on $C_B$ by $\phi$ (our statement $M[X] = N[Y]$). The proof then can be interpreted as saying that the necessity of lines such as $L$ going onto similar lines imparts so much rigidity that the birational mapping must essentially take $\Gamma_A$ onto $\Gamma_B$.

(10) Let $A$ be a noetherian ring of transcendence degree one over a field and suppose $A$ has a unique minimal prime $p$ such that $A/p = k[t'] = k^{(1)}$. Then $A$ is invariant. Moreover the following conditions are equivalent:

1. $A$ is strongly invariant.
2. $A$ is not of the form $A_0[X] = A_0^{(1)}$ where $A_0$ is a local artinian ring.
3. $A$ has an embedded prime divisor of $(0)$.

Proof. First we show the equivalence of conditions (1), (2) and (3) above.

$(3) \Rightarrow (1)$. Let $p$ be the unique minimal prime of $A$. We must show that if $R = A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)}$ then $A^* = B^*$. Let $M$ be an embedded prime divisor of zero in $A$. Then $M$ is maximal, so there is a maximal ideal $N \subseteq B$ such that $N[Y_1, \cdots, Y_n] = M[X_1, \cdots, X_n]$. This is because $M[X_1, \cdots, X_n]$ is an associated prime of $(0)$ in $R$. For if $N_1, \cdots, N_k$ are the associated primes of $(0)$ in $B$, then $(N_i R)^k_{i=1}$ are the primes of $(0)$ in $R$. By the uniqueness of these primes, $M[X_1, \cdots, X_n] = N_i[Y_1, \cdots, Y_n]$ for some $i$. Now we go to

$$A^{(n)} = A^* = B*$$

and use almost exactly the same argument which concludes (8).

$(1) \Rightarrow (2)$. This is clear for $A_0[X] = A_0^{(1)}$ is not strongly invariant since $A_0[X][Y] = A_0[Y][X]$.

$(2) \Rightarrow (3)$. We must show that if $A$ has no embedded prime divisors of $(0)$, then $A$ is of the form $A_0[X] = A_0^{(1)}$. Let

$$A \xrightarrow{\sigma} A/p = k[t'] = k^{(1)}$$

be the natural map and let $I = \sigma^{-1}(k)$. Since $p$ is the nil radical of $I$, $I$ is a local ring with maximal ideal $p$. Thus $I$ is a complete local ring. By Cohen's structure theorem of complete local rings [ZS, Theorem 27,
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p. 304], I contains a field $k_0$ which is naturally isomorphic to $k$. Thus $k_0 \subseteq A$ and $\sigma(k_0)=k$. We now identify $k_0$ and $k$.

Choose a preimage $t$ of $t'$ in $A$. If $\lambda_1, \ldots, \lambda_r$ generate $p$, we have $k[\lambda_1, \ldots, \lambda_r][t]=A$. The element $t$ is, in fact, transcendental over $k[\lambda_1, \ldots, \lambda_r]=A_0$. For consider the homomorphism

$$k[\lambda_1, \ldots, \lambda_r][X] \xrightarrow{\phi} k[\lambda_1, \ldots, \lambda_r][t] = A.$$

We claim this is an isomorphism. Let $K=\ker \phi$. Then $K$ must be contained in $N(A_0)[X]=(\lambda_1, \ldots, \lambda_r)[X]$. If not, it would follow that there is a $f(t) \in p$ where $j$ is a nonzero polynomial with coefficients in $k$. But since $A/p=k[t']$ and $t'$ is transcendental over $k$, this is impossible. We suppose there is a least integer $m$ such that the kernel of the map $\sigma^m$ in the following commutative diagram is zero. Here the rows are exact.

$$
\begin{array}{ccc}
0 & \rightarrow & K \\
\downarrow & & \downarrow \\
0 & \rightarrow & k[\lambda_1, \ldots, \lambda_r][X] \\
\downarrow & & \downarrow \\
0 & \rightarrow & k[\lambda_1', \ldots, \lambda_r'][X] \\
\downarrow & & \downarrow \\
0 & \rightarrow & k[\lambda_1'', \ldots, \lambda_r''][X'''] \\
\end{array}
$$

We claim $K'=0$ (for convenience we assume $N(A)^m+1=0$). Now $N(A)^m$ is a finite module over $A/N(A)=k[t']$. Moreover this is a torsion free module since if for some $m \in N(A)^m$, $f \in k[t']$, $fn=0$ then $f(t)$ would be a zero divisor in $A$. However $A$ has no embedded primes of zero, and it follows that $f(t) \in p$. This implies $f(t')=0$. Thus there is a free basis for $N(A)^m$ over $A/N(A)$, say $\xi_1, \ldots, \xi_r$. Let $h \in K'$. Then $h=\sum \xi_i h_i(X)$ by the minimality of $m$. But $\sigma^r(h)=\sum \xi_i h_i(t')=0$ implies $h_i=0$ for each $i$ and thus $h=0$. Thus $\sigma^r$ is an isomorphism in contradiction of the minimality of $m$. It follows then that $\sigma$ is an isomorphism modulo every power of $N(A)$. But for a suitably high power, this is zero. Thus $\sigma$ is an isomorphism, and hence $(2) \Rightarrow (3)$.

To complete our proof we must show that $A_0[X]=A_0^{(1)}$ is invariant if $A_0$ is an artinian local ring. Suppose $A_0=k[\lambda_1, \ldots, \lambda_r]$ where the $\lambda_i$ are nilpotent and let $A=A_0[X]$. If $R=A^{(n)}=A[X_1, \ldots, X_n]=B[Y_1, \ldots, Y_n]=B^{(n)}$, then we may assume $B^*=k[t']=k^{(1)}$ since $k[X]$ is invariant [AEH]. Let $t$ be a preimage of $t'$ and

\[ \lambda_i = \eta_i + \sum Y_i g_i(Y_1, \ldots, Y_n) \in B[Y_1, \ldots, Y_n]. \]

Then since $N(R)=(\lambda_1, \ldots, \lambda_r)$ it follows that $(\eta_1, \ldots, \eta_r)$ generates $N(B)$ and $B=k[\eta_1, \ldots, \eta_r, t]$. If $h$ is a polynomial with coefficients in
k and \( h(\lambda_1, \cdots, \lambda_r) = 0 \), then substituting from (*) we see \( h(\eta_1, \cdots, \eta_r) = 0 \). Thus there is a well defined \( k \)-homomorphism \( k[\lambda_1, \cdots, \lambda_r] \to k[\eta_1, \cdots, \eta_r] \) which takes \( \lambda_i \to \eta_i \). This can be extended to a homomorphism of \( A \) onto \( B \). Thus \( A \) is invariant by (6).

(11) Theorem. If \( A \) is a one dimensional affine ring over a field \( k \), then \( A \) is invariant. Moreover \( A \) is strongly invariant unless \( A \) can be expressed in the form \( A_1 \oplus A_0[X] \) where \( A_0 \) is a local artinian ring and \( X \) is an indeterminate over \( A_0 \).

Proof. Let \( e_1, \cdots, e_r \) be a maximal set of pairwise orthogonal idempotents of \( A \). Then \( A = Ae_1 \oplus \cdots \oplus Ae_r \). Now apply (7), (8) and (10) to complete the argument.

(12) Remark. Our Definitions (1) and (2) of invariant and strongly invariant could be taken to define \( n \)-invariant and \( n \)-strongly invariant where \( n \) is the number of variables. We do not know if it is possible for a ring to be \( n \)-invariant and not \( m \)-invariant for different integers \( m \) and \( n \). In particular does \( 1 \)-invariant imply \( n \)-invariant?

We close with an example of a strongly invariant, one dimensional affine ring \( A \) such that there is an isomorphism of polynomial rings \( \sigma: A[X_1, \cdots, X_n] \to B[Y_1, \cdots, Y_n] \), yet \( \sigma(A) \neq B \). Let \( k \) be a field of characteristic 2 and let \( A = k[a^2, a^3, \theta] / (\theta^2) \). If \( A^{(n)} = A[X_1, \cdots, X_n] = B[Y_1, \cdots, Y_n] = B^{(n)} \) for some ring \( B \), then \( A^* = B^* \) since \( A^* = k[a^2, a^3] \) is a strongly invariant ring [AEH]. Now let \( X \) be an indeterminate over \( A \) and set \( B = k[a^2, a^3 + \theta X, \theta] \). Then \( A[X] = B[X] \) and \( A \neq B \). However we must show that \( X \) is an indeterminate over \( B \). If not, suppose \( b_0 + b_1X + \cdots + b_sX^s = 0 \) with \( b_i \in B \). Then since \( A^* = B^* \), we must have \( \theta \) divides each \( b_i \), say \( b_i = \theta g_i(a^2, a^3 + \theta X, \theta) \). We have then

\[
0 = \sum_{i=0}^{s} \theta g_i(a^2, a^3 + \theta X, \theta)X^i = \sum_{i=0}^{s} \theta g_i(a^2, a^3, \theta)X^i
\]

since \( \theta^2 = 0 \). Hence \( \theta g_i(a^2, a^3, 0) = 0 \) for each \( i \) since \( X \) is transcendental over \( A \). Therefore \( g_i(T, Y) \equiv 0 \mod(T^3 - Y^2) \) so

\[
g_i(T, Y, Z) = (T^3 - Y^2)h(T, Y, Z) + Zq(T, Y, Z)
\]

so

\[
g_i(a^2, a^3 + \theta X, \theta) = [(a^2)^3 - (a^3 + \theta X)^3]h + \theta q = \theta q.
\]

Hence \( b_i = \theta g_i(a^2, a^3 + \theta X, \theta) = \theta \theta q = 0 \) which shows \( X \) is transcendental over \( B \).

Added in proof. M. Hochster has recently given an elegant example of an integral domain which is not invariant. Hochster's example is a four dimensional affine ring over the field of real numbers.
REFERENCES


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