MULTIVALUED OPERATIONS AND
UNIVERSAL COALGEBRA

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ABSTRACT. We define a type of representation of a semigroup by relations on a set, more general than the representation by transformations, which leads to a category cotripleable over the category of sets. This result motivates a generalization to higher-order operations and a concept of cotheory resembling that of theory in universal algebra.

1. Introduction. The representation of a semigroup $M$ by transformations of a set $A$ amounts to a homomorphism of $M$ into the semigroup of functions $A \rightarrow A$. The (left) $M$-sets are the objects of a category which is cotripleable over the category $\mathcal{S}$ of sets, the right adjoint $F$ being given by $FA = A \times A^M$ for each set $A$. Here $M$ acts on $FA$ by $m(a, h) = (h(m), mh)$ where $(mh)(n) = h(nm)$. In particular, $FA$ is universal in that any representation of $M$ on $A$ is embeddable in $FA$.

The relations on $A$ also form a semigroup $\mathcal{R}(A)$, whose elements can be thought of as functions $m: A \rightarrow PA = \text{power set of } A$. Composition is defined by $(mn)a = \bigcup \{ma' | a' \subseteq na\}$. We can also represent $M$ by relations on $A$, i.e. by a homomorphism $M \rightarrow \mathcal{R}(A)$. The category of $M$-sets in this sense does not admit a right adjoint, in general; this may be proved as in [2].

In this paper we exhibit a notion of representation lying between these two in generality which yields a cotripleable category. This will appear fortunate in view of the fact that the new kind of representation is not equivalent to a homomorphism from $M$ into any fixed semigroup connected with $A$.

The cotripleable categories so obtained admit an immediate generalization to categories of algebras over certain "cotheories" roughly analogous to the theories of universal algebra [4]. As noted in [2], the results of [1] imply that universal coalgebra has a notion of theory which is in principle exactly analogous to the one in universal algebra, but in practice is
awkward. This fact prompts a search for other characterizations of cotripleable categories. The concept introduced here appears to point in a promising direction.

2. Multivalued $M$-sets. If a semigroup $M$ acts on a set $A$ in the usual sense then $m(na) = (mn)a$. One way to interpret this equation when $M$ is represented by many-valued functions (relations) is that $ma' = (mn)a$ for each $a'$ in $na$; $(mn)a$ would be empty if $na$ is. A representation satisfying this requirement will be called a multivalued representation, or $\mu$-representation. A relation $R$ on $A$ can be in the image of a $\mu$-representation of some $M$ iff $R(a, b)$ and $R(a, c)$ imply that $\{x | R(b, x)\} = \{x | R(c, x)\}$; for, if $R$ satisfies this condition, the cyclic subsemigroup of $\mathcal{R}(A)$ generated by $R$ will be $\mu$-represented. (Here, as usual, $R \subseteq A \times A$ corresponds to $m : A \to PA$ via $b \in ma$ iff $R(a, b)$.) A relation $R$ satisfying this condition will be called a $\mu$-relation. However, the $\mu$-relations are not closed under composition and do not form a subsemigroup of $\mathcal{R}(A)$. Thus, for example, let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 3), (2, 4), (3, 4)\}$ and $S = \{(2, 1), (2, 4), (1, 3), (4, 3)\}$. Then $R$ and $S$ are $\mu$-relations but $RS = \{(1, 4), (2, 2), (2, 3), (4, 4)\}$ is not a $\mu$-relation. In the face of this fact, the right adjoint $F$ to be introduced below provides a way to get hold of the $\mu$-representations: a $\mu$-representation of $M$ on $A$ is a subalgebra of $FA$.

If $M$ has a $\mu$-representation on both $A$ and $B$, a function $f : A \to B$ will be called a homomorphism if for each $a$ and $m$, $mf(a) = f(ma)$ (direct image). It is straightforward to check that the evident forgetful functor from the resulting category $\mathcal{M}$ to $\mathcal{S}$ satisfies the precise cotripleableness condition. This means that if $f, g : A \to B$ in $\mathcal{M}$ and we have in $\mathcal{S}$ a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{s} & A \\
\downarrow{z} & & \downarrow{f} \\
B & \xrightarrow{g} & A \\
\end{array}
\]

where $sz = 1_E$, $tf = 1_A$, $tg = zs$, and $fz = gz$, then there is a unique $\mu$-representation of $M$ on $E$ making $z$ a homomorphism, and $z$ is then the equalizer of $f$ and $g$ in $\mathcal{M}$. Hence $\mathcal{M}$ is cotriple iff the forgetful functor has a right adjoint; see [1] or [3] for details.

In case $M$ is commutative we can exhibit the right adjoint $F : \mathcal{S} \to \mathcal{M}$ explicitly. For each set $A$ let $FA = A \times (PA)^M$ and define for each $n$ in $M$, $n(a, (a_m)_{m \in M}) = ((a', (a_{nm})_{m \in M}) | a' \in a_n)$. Then $FA$ is an object of $\mathcal{M}$, since

\[
(rn)(a, (a_m)_{m \in M}) = \{(a'', (a_{rnm})_{m \in M}) | a'' \in a_{rn}\} = \{(a'', (a_{nrnm})_{m \in M}) | a'' \in a_{nr}\} = r(a', (a_{nm})_{m \in M})
\]

for each $a'$ in $a_n$. Furthermore, if $B$ is in $\mathcal{M}$ and $f : B \to A$ is any function,
define \( g: B \rightarrow FA \) by \( g(b) = (f(b), (f(mb))_{m \in M}) \). Then

\[
g(nb) = \{g(b') \mid b' \in nb\} = \{(f(b'), (f(mb'))_{m \in M}) \mid b' \in nb\} = \{(f(b'), (f((mn)b))_{m \in M}) \mid b' \in nb\} = ng(b),
\]
because of the \( \mu \)-condition on \( B \), so \( g \) is a homomorphism and is easily seen to be unique with the property that \( p_Ag = f \). This proves the following result.

**Theorem 1.** If \( M \) is a commutative semigroup, the category of all sets equipped with \( \mu \)-representations of \( M \) is cotripleable over the category of sets.

It will follow from Theorem 2 that the word "commutative" is not necessary for the truth of Theorem 1.

3. **Cotheories.** The preceding would have gone through with little change if we had taken \( m: A \rightarrow P^2A = PPA \) and interpreted \( m(\alpha a) = (mn)a \) to mean that \( a' \in a \alpha a \) implies \( ma' = (mn)a \). This suggests the following analogy with the notion of an algebraic theory in universal algebra. Define a cotheory to be a category \( T \) with objects \( 0, 1, 2, \cdots \), distinguished maps \( T^n+1: n \rightarrow n+1 \), and a functor \( P: T \rightarrow T \) where \( P(n) = n+1 \). An algebra over \( T \) is to be a functor \( X: T \rightarrow \mathcal{S} \) such that \( X(n) = P^nA \) for a fixed set \( A \), \( XP = PX \), and \( X(T^{n+1}) \) takes \( \alpha \) to \( \{\alpha\} \). We denote \( X(\sigma) \) by \( \sigma_A \) for each \( \sigma \) in \( T \) and say that \( A \) is an algebra. The reasonable condition for a homomorphism \( f: A \rightarrow B \) is that for each \( \sigma: n \rightarrow m \), \( (P^m\sigma)(\sigma_A(\alpha)) = \sigma_B(P^n\sigma)(\alpha) \) where \( P \) is covariant (direct image). By \( P^nA \) we shall mean \( A \). If \( T \) is generated by maps \( 0 \rightarrow m \), we shall say that \( T \) is standard. This seems to be the correct generalization of multivalued operations, and is of significance since the category \( \mathcal{A} \) of algebras over a standard cotheory is easily seen to be right complete, and \( U: \mathcal{A} \rightarrow \mathcal{S} \) preserves colimits and satisfies the precise cotripleableness condition. Hence, by the adjoint functor theorem [5], \( \mathcal{A} \) is cotripleable iff \( U \) satisfies the cosolution set condition.

We have been able to obtain cosolution sets only when an additional strong condition is imposed on \( T \). If \( n > m \), define \( t^m_n = t^{n-1}_m \cdots \sigma t^{m+1}_m \). Suppose that, whenever \( \theta: 0 \rightarrow m \) and \( \sigma: 0 \rightarrow n \) in \( T \), there is \( \theta \sigma: 0 \rightarrow m \) such that the equation \( (P^n\theta)\sigma = t^{m+n}_m(\theta \sigma) \) holds in \( T \). Such a standard cotheory will be called tractable. Now, for each set \( A \), define

\[
FA = A \times \prod_{n=0}^{\infty} (P^nA)^{T(0,n)}.
\]
If $\sigma: 0 \to n$, define

$$\sigma_{f,A}(a, (x_0)) = \{ \cdots \{ (a', (x_{g+a})) \mid a' \in x_1 \} \cdots \mid x_{n-1} \in x_n \}.\]

Unfortunately the equations in $T$ need not hold in $FA$. However, if $B$ is a $T$-algebra and $f: B \to A$ is a function, we can define $g: B \to FA$ by

$$g(b) = (f(b), ((P^n f) \theta_B(b)) \theta_{0 \to m})$$

and $g$ can be seen to be a homomorphism whose image is fortunately a $T$-subalgebra of $FA$. Hence the set of $T$-subalgebras of $FA$ is a cosolution set for $A$. We have obtained the following generalization of Theorem 1.

**Theorem 2.** The category of algebras over a tractable standard cotheory is cotripleable over $\mathcal{S}$.

**References**


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