GENERALIZED RAMSEY THEORY FOR GRAPHS. II. SMALL DIAGONAL NUMBERS
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Abstract. Consider a finite nonnull graph $G$ with no loops or multiple edges and no isolated points. Its Ramsey number $r(G)$ is defined as the minimum number $p$ such that every 2-coloring of the lines of the complete graph $K_p$ must contain a monochromatic $G$. This generalizes the classical diagonal Ramsey numbers $r(n, n) = r(K_n)$. We obtain the exact value of the Ramsey number of every such graph with at most four points.

1. A celebrated Putnam question. The following question (see [3]) was already well known to most of those who knew it. Independently, it found its way into a Putnam examination where it attracted much attention:

"Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other."

Stated in the natural language [5] of graph theory, this asserts that whenever each of the 15 lines of the complete graph $K_6$ is colored either green or red, there is at least one monochromatic triangle.

Actually, there are at least two such triangles, as proved by Goodman [3]. Since we cannot color the lines of a graph green and red, we use solid and dashed lines instead in all the figures.

We proposed in [1] the more general approach of 2-coloring the lines of any graph $G$ and investigating whether there must occur a monochromatic copy of a specified subgraph $F$. Henceforth, a 2-coloring of $G$ will mean a coloration of the lines of $G$ with the two colors green and red.

A simple example (Figure 1) illustrating this viewpoint is obtained when we set $G = C_5$ and $F = P_3$. Whenever one colors the five lines of $C_5$ with two colors, there must obviously occur a monochromatic $P_3$.

\[ C_5:\] 
\[ P_3:\]

Figure 1.
2. The diagonal Ramsey numbers. The diagonal Ramsey number \( r(n, n) \) is defined [5, p. 16] as the smallest \( p \) such that in any 2-coloring of the complete graph \( K_p \), there always occurs a monochromatic \( K_n \).

Generalizing this concept, we now define the Ramsey number \( r(F) \) for any graph \( F \) with no isolated points. The value of \( r(F) \) is the smallest \( p \) such that in every 2-coloring of \( K_p \), there always occurs a monochromatic \( F \).

(This definition of \( r(F) \) coincides with that of \( r(F, 2) \) introduced in [2].) In particular, we have \( r(n, n) = r(K_n) \), and trivially \( r(K_2) = 2 \). The Putnam problem mentioned above amounts to showing that \( r(K_3) \leq 6 \). In fact, \( r(K_3) = 6 \) because the ten lines of \( K_5 \) can be colored green and red in such a manner that no monochromatic \( K_3 \) occurs. There is only one such 2-coloring (Figure 2), namely that which gives rise to a red \( C_5 \) and a green \( C_5 \) (pentagon and pentagram).

![Figure 2.](image)

Greenwood and Gleason [4] proved that \( r(K_5) = 18 \) by (a) producing a 2-coloring of \( K_{17} \) which has no monochromatic \( K_5 \), and (b) showing elegantly that every 2-coloring of \( K_{18} \) does contain such a \( K_5 \). Although upper and lower estimations for \( r(K_n) \) are known, the exact values of \( r(K_n) \) with \( n \geq 5 \) are still entirely open. Thus the determination of \( r(F) \) for the graphs with at most four points would bring us just up to \( r(K_5) \). It is our object to calculate \( r(F) \) exactly for these small graphs.

3. All stars. The Ramsey numbers of the stars are

\[
\begin{align*}
r(K_{1,m}) &= 2m, & m \text{ odd}, \\
               &= 2m - 1, & m \text{ even}.
\end{align*}
\]

(1)

We first prove (1) for odd \( m \). In this case, there is a regular graph \( G \) of degree \( m - 1 \) having \( 2m - 1 \) points, so its complement \( G' \) is regular of degree \( m - 1 \). Hence the decomposition (2-coloring) of \( K_{2m-1} \) into \( G \) and \( G' \) shows that \( r(K_{1,m}) \geq 2m \). The equality holds for in any 2-coloring of \( K_{2m} \), the green and red degrees of each point \( u \) sum to \( 2m - 1 \), whence one of these degrees is at least \( m \).
When $m$ is even, if there is a 2-coloring of $K_{2m-1}$ without a monochromatic star $K_{1,m}$, then both the green and red degree of each point equal $m-1$. But then the green graph is regular of degree $m-1$, which is a contradiction as both $m-1$ and $2m-1$ are odd. Thus we have $r(K_{1,m}) \leq 2m-1$. The equality follows from a decomposition of $K_{2m-2}$ into $G$ and $G$, where $G$ is a regular graph of degree $m-1$ with $2m-2$ points.

4. Small generalized Ramsey numbers. There are exactly ten graphs $F$ (Figure 3) with at most 4 points, having no isolates. We now find $r(F)$ for each of these. For convenience in identifying them, we use the operations on graphs from [5, p. 21], to get a symbolic name for each.

We have already seen that $r(K_2)=2$, $r(K_3)=6$ and $r(K_4)=11$. Setting $m=2$ and $m=3$ in (1), we obtain $r(K_{1,2})=3$ and $r(K_{1,3})=6$. Thus there are just five more graphs to investigate: $2K_2$, $P_4$, $C_4$, $K_{1,3}+x$ and $K_4-x$.

$r(2K_2)=5$. There is a 2-coloring of $K_5$ (Figure 4) with no monochromatic $2K_2$. On the other hand, it is ridiculously simple to verify that there is no such 2-coloring of the cycle $C_5$, a fortiori of $K_5$.

$r(P_4)=5$. By coincidence, Figure 4 shows that $r(P_4)>4$. We now exploit the fact, just noted, that every 2-coloring of $K_5$ has a monochromatic $2K_2$. Let $u_1u_2$ and $v_1v_2$ be two independent green lines in $K_5$. While trying to avoid a green $P_4$, we must color all four lines $u_1v_1$, red, thus producing an all red $P_4$, namely $u_1v_1u_2v_2$.

$r(C_4)=6$. Luckily, Figure 2 shows that $r(C_4)>5$. 

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Now assume there is a 2-coloring of $K_6$ with no monochromatic 4-cycle, $C_4$. As we already have $r(K_3)=6$, there is a (say) green triangle $u_1u_2u_3$ in $K_6$. Let $v_1, v_2, v_3$ be the other points. From each $v_i$, there is at most one green line to this green triangle, for otherwise, we have a green $C_4$. We now show that from each $v_i$, there is exactly one green line to the triangle. If not, all three lines $u_iv_i$ are red. But then the fact that at least two lines $u_iv_2$ are red gives a red $C_4$, like $v_1u_2v_2u_3v_1$. Next we rule out the possibility that there is more than one green line from any $u_i$ to the $v_j$, as shown in Figure 5(a) for $u_2$. This is seen from the red lines in Figure 5(b) which are forced while trying to avoid a green $C_4$.

![Figure 5](image1)

Now we know that there are green lines in this $K_6$ which must look like Figure 6, with no other green $u_iv_j$ lines.

![Figure 6](image2)

Clearly all the lines $v_iv_j$ are red. And now we have got it, because $v_1v_2v_3u_2v_1$ is a red $C_4$.

$r(K_{1,3}+x)=7$. The 2-coloring of $K_6$ in which $2K_3$ is red and $K_{3,3}$ is green (Figure 7) shows that $r(K_{1,3}+x)>6$. To prove that $r(K_{1,3}+x)=7$, we will show that it is impossible to have a 2-coloring of $K_7$ without a monochromatic $K_{1,3}+x$. To begin, we know by $r(K_3)=6$ that $K_7$ has (say) a green $K_3$ with points $u_1, u_2, u_3$. Call the other points $v_1$ to $v_4$. To avoid an immediate green $K_{1,3}+x$, we need to color all 12 lines $u_iv_j$ red (obtaining a
red \( K_{3,4} \). Next to avoid a sudden red \( K_{1,3} + x \), all 6 of the lines \( v_i v_j \) must be green. But behold we have a green \( K_4 \), hence a fortiori a green \( K_{1,3} + x \).

\( r(K_4 - x) = 10 \). If one stumbles on the correct example quickly (we did not), it is not at all difficult to see that \( r(K_4 - x) > 9 \). This example, which we believe to be the unique correct 2-coloring of \( K_9 \), is given by taking the cartesian product \( K_3 \times K_3 \) of two triangles as the green subgraph. Figure 8 shows only the green lines; those which are absent are red. Clearly, neither \( K_3 \times K_3 \) nor its complement contains \( K_4 - x \).

We now prove that \( r(K_4 - x) = 10 \). Consider an arbitrary 2-coloring of \( K_{10} \). By (1), there is a monochromatic (say green) \( K_{1,5} \), or in other words a point \( u \) adjacent greenly to 5 points \( u_i \), \( i = 1 \) to 5. We can now ignore the other four points and concentrate on the 10 lines \( u_i u_j \). There are two possibilities. If there is a green \( P_3 \) on the points \( u_i \), say \( u_4 u_5 u_6 \), then these 2 lines together with the 3 lines \( u_j u_i, j = 1, 2, 3 \), form a green \( K_4 - x \). On the other hand, if there is no green \( P_3 \) on the \( u_i \), then there are at most two green lines \( u_i u_j \). But every red graph with 5 points and 8 lines must contain a red \( K_4 - x \), completing the proof.

5. Conclusions. The small generalized diagonal Ramsey numbers just established are summarized in the following table:

<table>
<thead>
<tr>
<th>( F )</th>
<th>( K_2 )</th>
<th>( P_3 )</th>
<th>( K_3 )</th>
<th>( 2K_2 )</th>
<th>( P_4 )</th>
<th>( K_{1,3} )</th>
<th>( C_4 )</th>
<th>( K_{1,3} + x )</th>
<th>( K_4 - x )</th>
<th>( K_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(F) )</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>5</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>10</td>
<td>18</td>
</tr>
</tbody>
</table>
The next paper [2] in this series derives exact values of the small generalized off-diagonal Ramsey numbers for the above graphs $F$. These are defined on pairs of graphs $F_1, F_2$ as the smallest $p$ such that any 2-coloring of $K_p$ contains either a green $F_1$ or a red $F_2$. In another sequel [6], all the explicit 2-colorings of $K_6$ with the minimum number (two) of monochromatic triangles are displayed.

REFERENCES

5. F. Harary, Graph theory, Addison-Wesley, Reading, Mass., 1969. MR 41 #1566.