SETS OF MULTIPLICITY AND DIFFERENTIABLE FUNCTIONS

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Abstract. The paper contains two theorems relating the fine structure of differentiable functions, in one or more dimensions, to the behavior of Fourier-Stieltjes transforms on sets that are small in various ways.

In this paper we prove two theorems on the transformation of certain sets, defined as follows. A set \( F \) in a metric space is an \( L \)-set if there are sequences \( e_k \to 0 \) and \( \delta_k \to 0 \), and for each \( k \) a decomposition \( E = \bigcup_i E_i \), wherein \( \text{diam}(E_i) \leq e_k \delta_k \), while \( d(E_i, E_j) \geq \delta_k \) for \( i \neq j \). For each compact \( L \)-set \( E \) of real numbers there is a function \( h \) of class \( C^1(\mathbb{R}, \mathbb{R}) \) with \( h'>0 \), so that \( h(E) \) is a Kronecker set ([2], [3]). The first theorem is a complement to this.

**Theorem I.** Let \( \omega \) be a monotone, positive function on \((0, \infty)\), and \( \omega(0+) = 0 \); let \( C^1_\omega \) be the set of functions \( \varphi \) in \( C^1 \) with \( \varphi' > 0 \), \( |\varphi'(a) - \varphi'(b)| \leq \omega(|a-b|) \) for all real \( a \) and \( b \). Then there is a compact \( L \)-set \( E \) so that \( \varphi(E) \) is an \( M_\omega \)-set for each \( \varphi \) in \( C^1_\omega \).

To prove the theorem we choose a sequence of positive numbers \((c_n)\) so that \( c_0 = 1 \), \( \omega(c_n) < n^{-2} \), and \( c_{n+1} < n^{-3}c_n \). We now construct finite sets \( F_n \) and \( E_n \); the peculiar construction of \( F_n \) is the main point in the argument. \( F_n \) is a sequence of \( n^2 \) elements

\[
x(m) = x(0) + m_2 c_n n^{-5/2}, \quad 1 \leq m \leq n^2.
\]

Here \( x(0) = -c_n - c_{n} n^{-5/2} \) so that \( x(1) = 0 \). \( E_n \) is then a union of translates of \( F_n \), say \( \bigcup_i (E_n + a_i) \). Then \( a_0 = 0 \), while \( a_{j+1} - a_j = n^2 c_n + n^{-1/2}c_n \). In different terms, the final term in each translate becomes \( x(0) \) in its successor to the right. The number of translates is to be \([c_{n-1} c_n n^{-15/8}]\) for \( n \geq 1 \).

(a) In \( F_n \) we have the inequalities

\[
c_n \leq x(m+1) - x(m) \leq (n^2 + n^{3/2})c_n < 2n^3 c_n.
\]

Thus \( E_n \) has diameter \( 2n^{-1/6}c_{n-1} \). The vector sum \( E = \sum_{n=1}^\infty E_n \) is then an \( L \)-set. (It is somewhat easier to verify that, for large enough \( r \), the subset...
$E = \sum_{n=1}^{\infty} E_n$ is an $L$-set; this would serve just as well.) In each $E_n$ we construct the uniform probability distribution $\mu_n$ and then the convolution product $\prod_1^n \mu_n$, a probability in $E$. To prove the theorem, we demonstrate in fact that

$$\lim_{u \to +\infty} \int \exp(iu\varphi(t))\mu(dt) = 0 \quad \text{for each } \varphi \text{ in } C^1_\alpha.$$

To each large $u$ we choose an $n$ and observe

$$\left| \int \exp(iu\varphi(t))\mu(dt) \right| \leq \sup_{s \in E_n} \left| \int \exp iu\varphi(t + s)\mu_n(dt) \right|.$$ 

Certain exponents $u$ are handled without using the special properties of $\varphi$; the remaining exponents require more careful estimation. (In our second theorem we exploit this idea by making all exponents of the first sort.)

(b) $n^{1/5}c_n^{-1} \leq u \leq n^{-1/5}c_n^{-1}$. We must estimate the sums $\sum_{E_n} \exp(iu\varphi(x+s))$, $s \in E_n$, uniformly. Now for successive elements $z_i$ and $z_{i+1}$ of $E_n$ we have

$$|z_{i+1} - z_i - c_n| \leq 3n^{-3/2}c_n = o(c_n),$$

$$|u\varphi(z_{i+1} + s) - u\varphi(z_i + s) - u\varphi'(s)c_n| = o(uc_n).$$

The last relation requires only that $\varphi'$ be bounded and uniformly continuous on an interval about $E$. Now $uc_n \leq n^{-1/5}$, and the linear length of the sequence $\{u\varphi(x+s), x \in E\}$ is asymptotically $u\varphi'(s)c_n^{-1}n^{-1/6} \geq \varphi'(s)n^{1/30} \to +\infty$. Thus this part of the argument can be concluded by geometrical reasoning concerning uniform distribution modulo $2\pi$.

(c) For the remaining exponents we define $n$ by the inequality $n^{-1/5}c_n^{-1} < u < (n+1)^{1/5}c_n^{-1}$. Again we have recourse to uniform distribution, but first we split $E_n$ into its constituents $a_i + F_n$, and then split $F_n$ into residue classes modulo $n$. Thus we are attempting to estimate the distribution of sequences

$$u\varphi(s + x(m)), \quad 1 \leq m \leq n, m \equiv r \mod n.$$ 

Writing $y(p) = x(r + np), 0 \leq p < n, 1 \leq r \leq n$, we have sequences $u\varphi(y(p)+s)$, $0 \leq p < n$. To these sequences we apply an inequality of van der Corput [1, pp. 71–73] and conclude that it will be sufficient to obtain the uniform distribution of the difference sequences

$$u\varphi(y(p + h) + s) - u\varphi(y(p) + s), \quad 0 \leq p < n - h,$$

for $h = 1, 2, 3, \ldots$. (This need not be uniform with respect to $h$.) Now $F_n$ has diameter $<4n^2c_n$ so

$$\varphi(y(p + h)) - \varphi(y(p) + s) = [y(p + h) - y(p)]\varphi'(y(0) + s)$$

$$+ 0(y(p + h) - y(p))\omega(n^2c_n), \quad |\theta| \leq 4.$$
The error term can be majorized by

$$4 \cdot 4nhc_n \cdot o(c_{n-1}) = O(n^{-1}h c_n) = o(u^{-1}).$$

Therefore the error term can be neglected, as can the factor $\varphi'(y(0)+)$ in the remaining argument. Then

$$y(p + h) - y(p) = hnc_n + [h^2n^2 + 2hn(r + pn)]n^{-5/2}c_n$$

$$= A(h, n, r) + 2hn^{-1/2}c_n p.$$

Here $A(h, n, r)$ depends only on the variables indicated. Thus

$$uy(p + h) - uy(p) = A' + 2uhn^{-1/2}c_n p,$$

with

$$un^{-1/2}c_n < (n + 1)^{1/5}n^{-1/2} \to 0,$$

$$un^{-1/2}c_n \cdot n > n^{1/2}n^{-1/5} \to +\infty,$$

and $h \geq 1$.

The last two relations suffice for our purpose, since $p$ assumes the values in $[0, n - 1]$. Thus the exceptional exponents $u$ are disposed of, and the proof is complete.

In our second theorem we consider all $C^1$ maps from a rectangle in $R^k$ to a Euclidean space $R^m (m \geq 1)$. All maps except a set of the first category transform a certain set of uniqueness onto a $M_0$-set.

The theorem does not require Baire’s theorem to demonstrate the existence of the $C^1$ map, since maps with polynomial coefficients can be written explicitly.

Let $S_1$ and $S_2$ be sets of positive integers, each containing segments of unbounded length, and highly disjoint in the following sense: to each $K$ the inequality $|s_1 - s_2| < K (s_i \in S_i)$ has only a finite number of solutions $s_1, s_2$. Then $E_i$ is the set of sums $\sum_{n \neq s_i} e_n 2^{-n}$ ($e_n = 0, 1$), and so $E_i$ is an $L$-set. In $E_i$ we place the canonical product measure and on $E = E_1 \times E_2$ the measure $\mu = \mu_1 \times \mu_2$.

**Definition.** A measurable function $h$ on $E_1 \times E_2$ to $R^m$ is called projectively diffuse provided $\mu \{z : h(z) \in V\} = 0$ for every vector subspace $V \neq R^m$. Equivalently, $h$ is projectively diffuse provided

$$\lim_{\varepsilon \to 0} \mu \{z : |(h(z), u)| < \varepsilon \|u\|\}$$

uniformly for all $u$ in $R^m$.

**Theorem II.** (i) Let $F(x, y)$ be a $C^1$ map of $R^2$ into $R^m$ such that $\partial F / \partial x$ and $\partial F / \partial y$ are projectively diffuse. Then $F(E)$ is an $M_0$-set in $R^m$,

$$\lim \int \exp i(u, F) \, d\mu = 0 \quad \text{as} \quad \|u\| \to \infty \text{ in } R^m.$$
Moreover, these mappings form a set of second category in the B-space $C^1(I; \mathbb{R}^m)$, where $I$ is a closed rectangle containing $E$, and $\mu$ is an arbitrary diffuse measure on $E$.

It is easy to write down functions $F$, relative to measures $\mu = \mu_1 \times \mu_2$, provided only that each factor is a diffuse measure. Let $e_1, \ldots, e_m$ be a basis for $\mathbb{R}^m$ and let $F(x, y) = \sum e_i (x+y)^i$. Then for any linear form $f \neq 0$, $l(\partial F/\partial x) = l(\partial F/\partial y)$ has only a finite number of zeroes on any line, so its zero-set is $\mu_1 \times \mu_2$-null.

**Proof of Theorem II (i).** Let $E_n$ denote any of the subsets of $E$ determined by a choice of the first $n$ coordinates in the factors $E_1$ and $E_2$, and let $Q_n$ denote the closed convex hull of $E_n$. Then to each $\delta > 0$ there is an $\epsilon > 0$ and integer $N$ with this property: for every element $u$ of norm 1 in $\mathbb{R}^m$, the squares $Q_N$ meeting the set $\{ |(\partial_x F, u)| < \epsilon \}$ have total $\mu$-measure at most $\delta$. We call the remaining rectangles $Q_N$ admissible for $u$; they are disjoint except for a set of $\mu$-measure 0. Now let $u = \|u\| \eta_0$; to prove Theorem II (i) it suffices to prove that

$$\lim_{Q_N} \exp i(\eta_0, F) d\mu = 0, \quad Q_N \text{ admissible for } \eta_0.$$ 

Indeed, for each $\eta_0$ the admissible rectangles form a disjoint family of total measure $> 1 - \delta$. The restriction of $\mu$ to $Q_N$ is easily described; let $\mu_1^N$ and $\mu_2^N$ be the product measures on $\{ x \in \mathbb{R}^2 : n \in S_i, n > N \}$ and $\lambda^N = \mu_1^N \times \mu_2^N$. Then the restriction is obtained from $\lambda^N$ by a translation and a scalar multiplication. Therefore our problem is reduced to estimating integrals

$$\int \exp i \|u\| (\eta_0, F(z + z^*)) \lambda^N(dz), \quad z^* \in Q_N.$$ 

Suppose for definiteness that $\log \|u\| / \log 2$ is further from $S_1$ than from $S_2$. The integral is reduced to an iterated integral

$$\iint \exp i \|u\| G(x, y) \mu_1^N(dx) \mu_2^N(dy),$$

where $|\partial_x G| > \epsilon$ on a rectangle containing the support of the measure, and the metric properties of $\partial_x G$ are no worse than those of $\partial_x F$. Now $\mu_1^N$ contains as a factor the uniform distribution on a set $\{ \sum e_n 2^{-n} : r \leq n \leq p \}$, namely an arithmetic progression of difference $2^{-r}$, and $2^{p-r+1}$ terms. We can attain $\log \|u\| / \log 2 \to +\infty$, $p - \log \|u\| / \log 2 \to +\infty$, that is $2^{-r} \|u\| \to 0$, $2^{-r} \|u\| \to 0$. Since the progression has length on (the real line) $2^{-r+1}$, the proof can now be completed as in Theorem I.

**Proof of Theorem II (ii).** For any diffuse measure $\mu$, let $N_\mu$ be the set of $F$ in $C^1(I; \mathbb{R}^m)$ for which there is a linear form $l \neq 0$ so that
\[ \mu(z : \partial F(z) = 0) > 0; \] thus \( N_\mu \) is an \( F_\sigma \) in \( C^1 \). Further, if \( \mu = \sum_{j=1}^\infty \lambda_j \) is expressed as a sum of positive measures, then \( N_\mu = \bigcup_j N_{\lambda_j} \). Polynomials \( P(x, y) \) are dense in \( C^1 \) and by the device used after the statement of the theorem, we see that the set \( \{ p : \partial_x p \neq 0 \text{ for all forms } \neq 0 \} \) is dense in \( C^1 \).

Then \( N_\mu \) has void interior in \( C^1 \) unless \( \mu(Z) > 0 \), \( Z \) being the zero-set of some polynomial \( p \neq 0 \). Writing \( \mu(X) = \mu(X \cap Z) + \mu(X \setminus Z) = \lambda(X) + \lambda_\sigma(X) \), we can iterate this for the measure \( \lambda_2, \ldots \). Thus \( \mu = \lambda + \sum_{j=1}^\infty \lambda_j \), where \( N_\lambda \) has void interior and \( \lambda_j \) is concentrated on a zero-set \( Z_j \).

Next we observe that by the implicit function theorem each \( Z_j \) is a finite or countable union of analytic images of \((0, 1)\); we can therefore conclude that a polynomial having an uncountable number of zeroes on \( Z_\lambda \) vanishes identically on each connected component of \( Z_j \). To any \( m \) distinct points on an infinite component, \( Z_k \), there is a dense set of polynomials \( p \) so that \( \{ \partial_x p(Z_k) \}_1^m \) has rank \( m \) and hence for any \( l \neq 0 \), \( l(\partial_x p) \) has only isolated zeroes on the component. Because \( Z_j \) is a countable union of its components, \( \lambda_j \) is a sum of measures for which the exceptional sets are non-dense, and the theorem is proved.

REFERENCES


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