DIRICHLET FINITE SOLUTIONS OF $\Delta u = Pu$

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Abstract. The purpose of this paper is to give a necessary and also a sufficient condition for a Dirichlet finite harmonic function on a Riemann surface to be represented as a difference of a Dirichlet finite solution of $\Delta u = Pu$ ($P \geq 0$) and a Dirichlet finite potential of signed measure.

1. Let $P = P(z) \, dx \, dy$ ($z = x + iy$) be a nonnegative not identically zero $\alpha$-Hölder continuous ($0 < \alpha \leq 1$) second order differential on a Riemann surface $R$ and $PD(R)$ be the Hilbert space of all Dirichlet finite solutions of

$$\Delta u(z) = P(z)u(z), \quad \Delta \cdot = 4\partial^2 \cdot / \partial x \partial y,$$

on $R$ with the scalar product given by mixed Dirichlet integral, i.e.

$$(u, v) = DR(u, v) = \int_R du \wedge dv,$$

not the energy integral.

The study of $PD(R)$ was begun by Royden [6]. We will use the fact shown by Nakai [2] that $PD(R)$ forms a vector lattice under the natural order in $PD(R)$. We also use the Glasner-Katz maximum principle [1] that the modulus of every function in $PD(R)$ takes its maximum on the Royden harmonic boundary. The recent result of Nakai [3] that $PBD(R)$ is dense in $PD(R)$ will not be made use of.

Let $\Delta(R)$ be the Royden harmonic boundary and $HD(R)$ be the class of Dirichlet finite harmonic functions on $R$. (For the basic materials from the Royden compactification and the class $HD(R)$ we refer to the monograph of Sario and Nakai [7].) One of the important problems in the theory of $PD(R)$ which is not fully developed yet is to describe the distribution of $PD(R)|\Delta(R)$ in $HD(R)|\Delta(R)$. We will prove a theorem which contributes to this question.

2. If $R$ is parabolic, then $PD(R) = \{0\}$ (cf. Royden [6]), which case offers no interest. Therefore we assume throughout the paper that $R$ is hyperbolic. Let $\tilde{M}(R)$ be the class of all Dirichlet finite Tonelli functions on $R$ and $\tilde{M}_\Delta(R)$ the subclass of $\tilde{M}(R)$ consisting of functions $f$ with $f|\Delta(R) = 0$ (cf. [7]). We then have the orthogonal decomposition

$$\tilde{M}(R) = HD(R) + \tilde{M}_\Delta(R),$$
and since PD(R)⊂tilde{M}(R), we can define an operator $T:PD(R)\rightarrow HD(R)$ characterized by

$$u - Tu \in \tilde{M}_A(R).$$

Using results cited in §1 we can show that $T$ is a vector space isomorphism from PD(R) onto

$$X_D(R) \equiv T(PD(R))$$

such that $u>0$ is equivalent to $Tu>0$ and $\sup_R |u| = \sup_R |Tu|$. Therefore the study of PD(R) can be reduced to that of $X_D(R)$ and for this reason we call $T$ the reduction operator for Dirichlet finite solutions. It can be seen that

$$u = Tu - \frac{1}{2\pi} \int_R G(\cdot, \zeta)P(\zeta)u(\zeta) \, d\xi \, d\eta \quad (\zeta = \xi + i\eta)$$

(cf. [3]). We will discuss when $h \in HD(R)$ belongs to $X_D(R)$.

3. Let $\Omega$ be a regular subregion of $R$. By the $P$-unit of $\Omega$ we mean the solution $e_\Omega$ of (1) on $\Omega$ with the continuous boundary values 1. The net $\{e_\Omega\}$ for every regular subregion $\Omega$ is decreasing and hence convergent to a solution on $R$:

$$e_R = \lim_{\alpha \rightarrow R} e_\Omega \geq 0$$

which we call the $P$-unit on $R$. The only bounded solution of (1) on $R$ is zero if and only if $e_R \equiv 0$ (Ozawa [5], Royden [6]). We describe $X_D(R)$ in terms of $\{e_\Omega\}$ and $e_R$ as follows:

**Theorem.** Suppose that $h \in HD(R)$. If $h \in X_D(R)$, then

$$D_R(e_R h) < \infty.$$  

Conversely if

$$\limsup_{\alpha \rightarrow R} D_\Omega(e_\Omega h) < \infty,$$

then $h \in X_D(R)$.

The proof will be given in §§4 and 7. The condition (4) is necessary for $h \in X_D(R)$ but not sufficient. In fact, let $R = \{|z| < 1\}$ and $P(z) = 4(1 + |z|^2)(1 - |z|^2)^{-2}$. Then $e_R \equiv 0$ and $X_D(R) = \{0\}$ (Royden [6]), while (4) is trivially valid for every $h \in HD(R)$. The condition (5) is sufficient for $h \in X_D(R)$ but not necessary. We exhibit an instructive example due to Nakai [4]. Let $R = \{|z| > 1\}$ and $P(z) = 1 + |z|^{-1}$. Consider $\Omega_n = \{|1 + n^{-1} < |z| < n\}$ ($n = 2, 3, \cdot \cdot \cdot$) which exhausts $R$ as $n \rightarrow \infty$. Denote by $e_n$ the $P$-unit on $\Omega_n$. The $P$-unit $e_R$ on $R$ is given by

$$e_R(z) = \left( e^{\int_1^\infty e^{-2t} \, dt} \right)^{-1} \cdot e^{|z| \cdot \int_1^\infty e^{-2t} \cdot t^{-1} \, dt}.$$
and a straightforward calculation shows that $e_R \in \text{PD}(R)$ and hence $1 \in \text{XD}(R)$. We also see that

$$e_n(z) = \alpha_ne^{iz} + \beta_ne^{iz} \int_{1}^{1+1/n} e^{-2it} \, dt$$

where

$$\alpha_n = e^{-n} - (e^{-n} - e^{-(1+1/n)}) \left( \int_{1+1/n}^{n} e^{-2it} \, dt \right)^{-1} \int_{1}^{n} e^{-2it} \, dt,$$

and

$$\beta_n = (e^{-n} - e^{-(1+1/n)}) \left( \int_{1+1/n}^{n} e^{-2it} \, dt \right)^{-1}.$$

However an easy but cumbersome computation shows that

$$D_R(1 \cdot e_n) = \mathcal{O}(n) \quad (n \to \infty)$$

and (5) is not valid for $h=1 \in \text{XD}(R)$. By Fatou's lemma, (5) implies (4). That the converse is not necessarily true is also seen from the above example.

4. Necessity of (4). Suppose $h \in \text{XD}(R)$. Since $\text{XD}(R)$ forms a vector lattice along with $\text{PD}(R)$, we may assume $h > 0$ to prove (4). Let $u \in \text{PD}(R)$ such that $h = Tu$ and let $\varphi = u - h$. We will prove (4) both for $u$ and $\varphi$.

We write $l(u) = (D_\Omega())^{1/2}$. By Green's formula

$$\|u(1 - e_\Omega)\|_\Omega^2 = -\int_\Omega u(1 - e_\Omega) \, d * d(u(1 - e_\Omega))$$

$$= -\int_\Omega u^2(1 - e_\Omega)^2 \, d + \int_\Omega u^2(1 - e_\Omega) e_\Omega \, d + 2 \int_\Omega u(1 - e_\Omega) \, d \wedge * d e_\Omega$$

$$\leq \int_\Omega u(1 - e_\Omega) \, d \wedge (1 - e_\Omega) \, d + 2 \int_\Omega (1 - e_\Omega) \, d \wedge * (d(e_\Omega u) - e_\Omega \, d).$$

Observe that $\int_\Omega u(1 - e_\Omega) \, d \wedge * d = -\int_\Omega d(u(1 - e_\Omega)) \wedge d * du$. By Schwarz's inequality

$$\|u(1 - e_\Omega)\|_\Omega^2 \leq \|u(1 - e_\Omega)\|_\Omega \|u\|_\Omega + 2 \|u\|_\Omega \|ue_\Omega\|_\Omega + 2 \|u\|_\Omega^2.$$ 

In view of $\|ue_\Omega\|_\Omega \leq \|u(1 - e_\Omega)\|_\Omega + \|u\|_\Omega$, we deduce

$$\|u(1 - e_\Omega)\|_\Omega^2 \leq 3 \|u(1 - e_\Omega)\|_\Omega \cdot \|u\|_\Omega + 4 \|u\|_\Omega^2.$$ 

This implies $\|u(1 - e_\Omega)\|_\Omega \leq 4 \|u\|_\Omega$ or $\|ue_\Omega\|_\Omega \leq 5 \|u\|_\Omega$. Therefore, by Fatou's lemma,

$$D_R(e_R u) \leq \liminf_{\Omega \to R} D_\Omega(e_\Omega u) \leq 25D_R(u) < \infty.$$
5. Let \( h_\Omega \in C(\Omega) \) such that \( h_\Omega \) is harmonic in \( \Omega \) and \( h_\Omega|_{\partial \Omega} = u \). Set \( \varphi_\Omega \equiv u - h_\Omega \). Observe \( \Delta \varphi_\Omega = Pu \) and \( \varphi_\Omega \leq 0 \). Since \( D_\Omega(u) = D_\Omega(h_\Omega) + D_\Omega(\varphi_\Omega) \) and \( \lim_{\Omega \to R} h_\Omega = h \), we infer that \( \varphi = \lim_{\Omega \to R} \varphi_\Omega \), \( d\varphi = \lim_{\Omega \to R} d\varphi_\Omega \), and \( D_\Omega(\varphi_\Omega) \leq D_\Omega(u) \). By Green's formula,

\[
\|e_\Omega \varphi_\Omega\|^2 = -\int_\Omega e_\Omega \varphi_\Omega d^*d(e_\Omega \varphi_\Omega) = -\int_\Omega e_\Omega^2 \varphi_\Omega^2 P - \int_\Omega e_\Omega \varphi_\Omega u P - 2 \int_\Omega e_\Omega \varphi_\Omega \, de_\Omega \wedge *d\varphi_\Omega \leq -\int_\Omega \varphi_\Omega d^*du - 2 \int_\Omega e_\Omega d\varphi_\Omega \wedge *(\varphi_\Omega \, de_\Omega) = \int_\Omega d\varphi_\Omega \wedge *du - 2 \int_\Omega e_\Omega d\varphi_\Omega \wedge *(d(e_\Omega \varphi_\Omega) - e_\Omega d\varphi_\Omega).
\]

By Schwarz's inequality,

\[
\|e_\Omega \varphi_\Omega\|^2 \leq \|\varphi_\Omega\|_\Omega \|u\|_\Omega + 2 \|\varphi_\Omega\|_\Omega \|e_\Omega \varphi_\Omega\|_\Omega + 2 \|\varphi_\Omega\|^2 \leq 2 \|u\|_\Omega \|e_\Omega \varphi_\Omega\|_\Omega + 3 \|u\|^2_\Omega
\]

and therefore \( \|e_\Omega \varphi_\Omega\|_\Omega \leq 3 \|u\|_\Omega \). By Fatou's lemma we deduce

\[
D_R(e_R \varphi) \leq \liminf_{\Omega \to R} D_\Omega(e_\Omega \varphi_\Omega) \leq 9D_R(u) < \infty.
\]

6. Sufficiency of (5). Let \( u_\Omega \in C(\Omega) \) such that \( \Delta u_\Omega(z) = P(z)u_\Omega(z) \) on \( \Omega \) and \( u_\Omega|_{\partial \Omega} = h \). By Green's formula,

\[
\|u_\Omega - he_\Omega\|^2 = -\int_\Omega (u_\Omega - he_\Omega)d^*d(u_\Omega - he_\Omega) = -\int_\Omega (u_\Omega - he_\Omega)u_\Omega P + \int_\Omega (u_\Omega - he_\Omega)he_\Omega P + 2 \int_\Omega (u_\Omega - he_\Omega) \, dh \wedge *de_\Omega \leq 2 \int_\Omega \, dh \wedge *(d(e_\Omega u_\Omega) - e_\Omega \, du_\Omega) - 2 \int_\Omega e_\Omega \, dh \wedge *(d(he_\Omega) - e_\Omega \, dh).
\]

By Schwarz's inequality,

\[
\|u_\Omega - he_\Omega\|^2 \leq 2 \|h\|_\Omega \|e_\Omega u_\Omega\|_\Omega + 2 \|h\|_\Omega \|u_\Omega\|_\Omega + 2 \|h\|^2_\Omega.
\]
By the same estimate as in §4, we deduce \( \|e_{\Omega}u_{\Omega}\|_{\Omega} \leq 5\|u_{\Omega}\|_{\Omega} \) and \( \|u_{\Omega}\|_{\Omega} \leq \|u_{\Omega} - he_{\Omega}\|_{\Omega} + \|he_{\Omega}\|_{\Omega} \), and hence
\[
\|u_{\Omega} - he_{\Omega}\|_{\Omega}^2 \leq 12\|h\|_{\Omega}\|u - he_{\Omega}\|_{\Omega} + 14\|h\|_{\Omega}\|he_{\Omega}\|_{\Omega} + 2\|h\|_{\Omega}^2.
\]
By (5) we conclude that
\[
D_{\Omega}(u_{\Omega}) \leq K < \infty
\]
for every \( \Omega \) with a constant \( K \).

7. Fix an \( \Omega_0 \) such that \( P \notin \Omega_0 \). Since \( PD(\Omega_0) \) is a Hilbert space with reproducing kernel (cf. [2]), (6) implies that there exists an exhaustion \( \{\Omega_n\} \) of \( \Omega \) with \( \Omega_{n+1} \supset \Omega_n \) such that \( \{u_{\Omega_n}\} \) converges uniformly on each compact set of \( \Omega_n \). By a diagonal process, we may assume that \( \{u_{\Omega_n}\} \) converges uniformly on each compact set of \( \Omega \). Let \( u = \lim_{n \to \infty} u_{\Omega_n} \). Because of (6) and Fatou’s lemma we see that \( u \in PD(\Omega) \). We can regard \( h - u_{\Omega_n} \) as an element of \( M_0(\Omega) \subset M_\Delta(\Omega) \). Since \( \lim_{n \to \infty} (h - u_{\Omega_n}) = h - u \) uniformly on each compact set of \( \Omega \) and \( \sup_n D_R(h - u_{\Omega_n}) < \infty \), Kawamura’s lemma (cf. [7]) implies that \( h - u \in M_\Delta(\Omega) \), i.e. \( h = Tu \) and a fortiori \( h \in X_D(\Omega) \).

REFERENCES

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