CYCLIC VECTORS OF A CYCLIC OPERATOR SPAN THE SPACE

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ABSTRACT. If a bounded linear operator on a Banach space has a cyclic vector, then the set of cyclic vectors spans the space.

In their paper [1], B. Sz.-Nagy and C. Foiaș proved, as a corollary of a result on unitary dilations, that if a (linear bounded) operator $T$ on a Hilbert space has a cyclic vector then the set of all cyclic vectors of $T$ spans the whole space.

Here we give an entirely elementary proof of this theorem which also applies to operators on Banach spaces.\(^1\)

THEOREM. Let $T$ be an operator on the Banach space $X$. Suppose $x_0$ is a cyclic vector for $T$. Then, for any constant $c \neq 0$ such that $\|cT\| < 1$, the vectors

$$x_p = (I - cT)^p x_0 \quad (p = 1, 2, \cdots)$$

are also cyclic for $T$. The vectors $x_p (p = 0, 1, \cdots)$ span $X$ linearly.

PROOF.\(^1\) (1) Since $\|cT\| < 1$, we have

$$I = (I - cT)^{-1} = \sum_{n=0}^\infty (cT)^n I - cT) \quad \text{(in operator norm)},$$

hence

$$x_p = \sum_{n=0}^\infty c^n T^n x_{p+1},$$

and consequently

$$T^m x_p = \sum_{n=0}^\infty c^n T^{n+m} x_{p+1} \quad (m = 0, 1, \cdots).$$

This shows that if $x_p$ is cyclic for $T$, then $x_{p+1}$ is cyclic for $T$ too. Thus all the $x_p$'s ($p \geq 0$) are cyclic.

(2) From the relation

$$(cT)^n = [I - (I - cT)]^n = \sum_{p=0}^n (-1)^p \binom{n}{p} (I - cT)^p \quad (n = 0, 1, \cdots),$$

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\(^1\) This proof was made in 1968 as mentioned in a footnote of [1].

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it follows

\[ T^n x_0 = \frac{1}{c^n} \sum_{p=0}^{n} (-1)^p \binom{n}{p} x_p \quad (n = 0, 1, \ldots). \]

Since the set \( \{T^n x_0\}_{n=0}^{\infty} \) spans \( X \) by virtue of the cyclicity of \( x_0 \), we infer that the set \( \{x_p\}_{p=0}^{\infty} \) also spans \( X \).

This finishes the proof.

**REFERENCE**


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