UNITARY GROUPS AND COMMUTATORS

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Abstract. If \( H \) is a possibly unbounded selfadjoint operator and \( A \) is a closed operator in a Hilbert space, the relation 
\[
(U_t^{-1}AU_t)f = iU_t^{-1}(AH-HA)U_t f
\]
can be shown to hold under relatively reasonable hypotheses on \( A \) and \( f \), where \( U_t = e^{iHt} \). This relation can then be used to relate properties of the commutator \( AH-HA \) to properties of \( A \) and \( H \).

In quantum mechanics, a state \( f \) at time \( t=0 \) evolves at time \( t_0 \) into the state \( U_{t_0}f \), where \( U_t = e^{iHt} \) and \( H \) is the quantum mechanical Hamiltonian operator for the system. This means that for the observable \( A \), the expectation of \( A \) in the state \( U_{t_0}f \) is given by \( \langle AU_{t_0}f, U_{t_0}f \rangle \). Equivalently, we may regard the state as fixed and the observable \( A \) as evolving with time. Thus at time \( t \) the new observable \( A_t \) is \( U_t^{-1}AU_t \). To analyze this evolution further, an obvious step is to differentiate with respect to \( t \), which yields the formal relation 
\[
A_t' = iU_t^{-1}(AH-HA)U_t.
\]
If \( i(AH-HA) \) is positive definite, for example, this means that expectations are increasing.

Thus one is naturally led to study the commutator \( AH-HA \). We shall use the group \( U_t \) as an essential tool in our study, and the hypotheses of our theorems will explicitly involve \( U_t \). This seems justified physically, since \( U_t \) has direct physical significance.

A quite different method of relating \( A, H \) and \( AH-HA \) is given in the interesting book by Putnam [3].

In what follows, we let \( U_t = e^{iHt} \), and \( H \) be a selfadjoint operator in a Hilbert space \( h \). \( A \) will be a closed operator in \( h \). Take domain \( H^n \) to mean the intersection of the domains of all \( H^n \), where \( n \) ranges over the positive integers. Take \( H^0 \) to be the identity operator.

We first state and prove conditions under which the relationship 
\[
(U_t^{-1}AU_t) = iU_t^{-1}(AH-HA)U_t
\]
holds.

Theorem 1. Let \( n \) be a nonnegative integer, and let \( m > n \) be a positive integer or \( \infty \). Suppose that domain \( A \) contains domain \( H^n \), and that \( A \) takes
domain $H^m$ into domain $H$. Then, for any $f$ in domain $H^m$, $(U_t^{-1}A U_t f)'$ exists in the strong sense and is equal to $i[U_t^{-1}(AH-HA)U_t]f$.

**Remark.** If $AH$ and $HA$ were both defined on domain $H^i$, for some nonnegative integer $i$, the hypotheses of Theorem 1 would hold, taking $n=i$, and $m=i+1$.

**Proof.** We prove the theorem by taking difference quotients, after first observing that $U_t$ takes domain $H^i$ onto itself, for any $i$ which is either a nonnegative integer or $\infty$.

Now
\[ U_{t+\Delta t}^{-1}A U_{t+\Delta t} f - U_t^{-1}A U_t f = U_t^{-1}[U_{\Delta t}^{-1}A U_{\Delta t} - A]U_t f. \]
Calling $U_t f = g$, we note that $g$ is in domain $H^m$. But
\[ (1/\Delta t)[U_{\Delta t}^{-1}A U_{\Delta t} - A]g = (1/\Delta t)(U_{-\Delta t} - I)Ag + (U_{-\Delta t}A(U_{\Delta t} - I)g)(1/\Delta t). \]
As $\Delta t$ approaches zero, the first term goes to $-iHAg$, since $Ag$ is in domain $H$ by hypothesis. The second term is a little harder to analyze.

First, we note that $A$ defines a closed, and therefore continuous linear transformation of $B$ into $h$, where $B$ is the Banach space created by giving domain $H^m$ the graph norm associated with $H^m$.

However $(U_{\Delta t} - I)g/\Delta t$ approaches $iHg$ in $B$ as $\Delta t$ approaches zero, since $g$ is in domain $H^{n+1}$. Thus $A(U_{\Delta t} - I)g/\Delta t$ converges to $iAHg$ in $h$.

But, finally, from strong continuity of $U_t$ and the fact that $\|U_t\| = 1$ for all $t$, it follows that $U_{-\Delta t}A(U_{\Delta t} - I)(g/\Delta t)$ approaches $iAHg$ as $\Delta t$ approaches zero.

Collecting what we have proved, we see that $(1/\Delta t)[U_{-\Delta t}^{-1}A U_{-\Delta t} - A]g$ approaches $i(AH-HA)g$ as $\Delta t$ approaches zero. From continuity of $U_t$ it follows that $(1/\Delta t)U_{-\Delta t}(U_{-\Delta t}^{-1}A U_{-\Delta t} - A)U_t f$ approaches $i(U_{-\Delta t}(AH-HA)U_t f$ in $h$ as $\Delta t$ approaches zero. This completes the proof of Theorem 1.

**Corollary 1.** Under the hypotheses of Theorem 1, it cannot happen that $(i(AH-HA)U_t f, U_t f) > C \|U_t f\|^2$ for any $C > 0$, and neither can it happen that $(i(AH-HA)U_t f, U_t f) < -C \|U_t f\|^2$ for all $t$ in any infinite interval.

**Proof.** By the closed graph theorem, and the fact that in the graph norm generated by $H$ on domain $H$, the norm of $U_t f$ is the same as that of $f$, it follows that $\|AU_t f\|$ remains bounded. Therefore so does $(AU_t f, U_t f)$. By Theorem 1, the proof is completed.

**Definition.** An operator $A$ is said to be local with respect to $U_t f$ if $U_t f$ is contained in domain $A$ for all $t$, and $AU_t f$ approaches zero as $t$ approaches $\pm \infty$. 
It might at first appear that it is hard to show that an operator $A$ is local with respect to $U_t f$. This is not the case, however, for many types of self-adjoint operators $H$ which are important in applications. A few remarks on this problem seem in order.

First, if $H$ is a self-adjoint operator in $\mathcal{H}$, and $f$ is an element of $\mathcal{H}$, then recall that $f$ is said to be absolutely continuous with respect to $H$ if the real valued measure $m_f(S)=\|P(S)f\|^2$ is absolutely continuous with respect to Lebesgue measure on $R$. Here $S$ is any borel set in $R$, and $P(S)$ is the projection associated with $S$ by the spectral measure associated with $H$. The set of all such $f$ forms a reducing subspace of $H$, and the restriction of $H$ to this subspace forms a self-adjoint operator $H_a$. $U_t$ takes this subspace into itself. If $H_a=H$, $H$ is said to be absolutely continuous.

Now, if $f$ is absolutely continuous with respect to $H$, $H$ is a self-adjoint ordinary differential operator and $h=L^2(R)$, it follows by an argument in Lax and Phillips [2, p. 147], that $\|C_\Delta U_t f\|$ approaches 0 as $t$ approaches $\pm\infty$, where $C_\Delta$ is the characteristic function of any compact interval $\Delta$. Here $H$ must be assumed to have order one or greater. Thus if $A$ is a bounded operator, and $A$ is the limit in operator norm of a sequence of operators $A_n$ defined by $A_n f=AC_{\Delta_n} f$ for a sequence of compact intervals $\Delta_n$, it follows that $A$ is local with respect to $U_t f$, provided $U_t=e^{iH_t}$, $H$ is a self-adjoint ordinary differential operator, and $f$ is absolutely continuous with respect to $H$. An example of such an $A$ is multiplication by a $C_0$ function.

It may be shown (see Kato [1]) that many ordinary differential operators have nontrivial absolutely continuous parts, and that therefore such vectors $f$ may be found. Further, similar considerations can be made to apply to the case where $A$ is an ordinary differential operator with $C_0$ coefficients, provided $H$ is a self-adjoint ordinary differential operator in $L^2(R)$ with bounded coefficients and nontrivial absolutely continuous part and $A$ is of order less than or equal to that of $H$.

Another way of showing locality, which also applies to differential operators is contained in the following theorem.

**Theorem 2.** Let $H$ be absolutely continuous. Let $A$ be $H$-compact. Then $A$ is local with respect to $U_t f$ for all $f$ in domain $H$.

**Remark.** Recall that $A$ is said to be $H$-compact if domain $A$ contains domain $H$, and $A$ is a compact operator from domain $H$ into $h$, where domain $H$ is equipped with the graph norm from $H$.

**Proof.** Since $H$ is absolutely continuous, then by the Riemann-Lebesgue lemma $(U_t f, g)$, which equals $(\int_{-\infty}^{\infty} e^{it\Delta} dP f, g)$, approaches 0 when $t$ approaches $\pm\infty$ for all $f$ and $g$ in $h$. Now suppose $A$ is not local.
with respect to some \( f \) in domain \( H \). Then there is a sequence \( t_n \) approaching, say, \( +\infty \) such that \( \|AU_{t_n}f\| > C \) for some \( C > 0 \). Since \( A \) is \( H \)-compact, it follows that, for some subsequence \( t_{n(i)} \), \( AU_{t_{n(i)}}f \) approaches \( g \), with \( \|g\| \geq C \).

Let \( U_{t_n}f = f_i \). Let \( \|Af_i\| \leq M \). Select \( g_1 \) in domain \( A^* \) such that \( \|g_1 - g\| \leq C^2/2M \). Then \( \|(Af_i, g_1) - (Af_i, g)\| \leq M \|g_1 - g\| \leq C^2/2 \). Therefore, when \( j \) is large enough, \( \|(f_j, A^*g_1)\| \geq C^2/4 \). Therefore \( |(f_j, A^*g_1)| \geq C^2/4 \), which contradicts the fact \( f_j \) converges weakly to 0. This completes the proof.

We now use the hypothesis of locality with respect to \( U_t f \).

**Theorem 3.** Suppose that \( AH^2, HAH \) and \( H^2A \) are all defined on domain \( H^{n-1} \) for some positive integer \( n \geq 3 \). Suppose \( A \) is local with respect to \( U_t f \), where \( f \) is in domain \( H^n \), and suppose there is a dense subspace \( S \) of \( H \) on which \( A^*H^2 \), \( HA^*H \) and \( H^2A^* \) are defined. Then \( AH - HA \) is local with respect to \( U_t f \).

**Remark.** If \( A \) is symmetric, the last hypothesis is obviously redundant. Also if \( A \) and \( H \) are ordinary differential operators, \( C^0 \) will usually be such a subspace \( S \).

**Proof.** \( \left( U_t^{-1}AU_tf\right)^* = U_t^{-1}(AH^2 - 2HAH + H^2A)U_t f \) as may be seen using Theorem 1. It is of course necessary to show that the operator \( AH - HA \), when restricted to domain \( H^{n-1} \), has a closed extension. However, an operator has a closed extension if and only if its adjoint is densely defined, so our last hypothesis takes care of this possibility.

Now if \( T \) is the operator \( AH^2 - 2HAH + H^2A \), restricted to domain \( H^n \), then \( T \) has a closed extension. Therefore, giving domain \( H^n \) the graph norm from \( H^n \), and letting \( \tilde{T} \) denote the operator induced by \( T \) from this Banach space into \( H \), we see that \( \tilde{T} \) is continuous by the closed graph theorem.

But since \( H^nU_t f = U_t H^n f \), it follows that all \( U_t f \) have the same norm as \( f \) in the graph norm on domain \( H^n \). Therefore the set of all \( U_t f \) is a bounded set in the Banach space domain \( H^n \), so that the set of \( TU_t f \) is a bounded set in \( h \). Therefore the set of all \( (U_t^{-1}AU_t f)^* \) is a bounded set in \( h \), as \( t \) ranges over the whole real line.

Let \( f'_t \) be \( U_t^{-1}AU_t f \). We need to show that \( f'_t \) approaches zero in norm, as \( t \) approaches \( \pm \infty \), in order to prove the theorem. Let \( g(t) = (f'_t, f_t) \). Then \( g'(t) = (f'_t, f'_t) + (f_t, f'_t) \). Also, \( g''(t) = (f''_t, f'_t) + 2(f'_t, f''_t) \). Since \( f''_t \) is bounded, and \( f_t \) goes to zero as \( t \) approaches \( \pm \infty \), it follows that \( (f''_t, f_t) \) also approaches zero.

To show that \( (f'_t, f'_t) \) approaches 0, we first observe that

\[
 f'_t = iU_t^{-1}(AH - HA)U_t f, 
\]
which, once again, by the closed graph theorem, remains bounded as $t$ approaches $\pm \infty$. Thus $(f'_n, f'_n)$, which equals $(f'_n, f'_n) + (f'_n, f'_n)$, is a bounded real valued function of $t$. If there were a sequence $t_n$ approaching, say, $+\infty$ such that $(f'_n, f'_n) > \varepsilon$, then there would have to be a $\delta$ such that $(f'_n, f'_n)$ remained $\geq \varepsilon/2$ on $[t_n - \delta, t_n + \delta]$ for all $n$, by the mean value theorem. Thus there would be an $N$ such that $g''$ would be greater than $\varepsilon/4$ on the interval $[t_n - \delta, t_n + \delta]$ for all $n \geq N$. However, since $f'_n$ approaches zero and $f'_n$ remains bounded as $t$ approaches infinity, it is clear that $g'(t)$ must approach 0. This contradicts the fact we just discovered about $g''$. The theorem is proved.

**Corollary 2.** Under the hypotheses of Theorem 3, it cannot happen that $AH - HA$ has a bounded inverse when restricted to the linear span of the $U_tf$.

**Corollary 3.** Let $f$ be as in Theorem 3, and suppose $f$ is perpendicular to the eigenvectors of $H$. Let $T$ be the operator formed by restricting $AH - HA$ to the linear span of the $U_tf$ and $T_1$ be the closure of the graph of $T$ in the product space $h \times h$. Then $T_1$ cannot be a linear operator with closed range and finite dimensional null space.

**Proof.** There is a sequence $t_n$ approaching infinity such that $U_t f$ approaches 0 weakly in $h$. (See Lax and Phillips [2, p. 145].) But $(AH - HA)U_t f$ approaches zero by Theorem 3.

Let $S$ be the closed linear span of the $U_tf$. If $T_1$ is a closed operator defined on a dense subspace of $S$, and $K$ is its null space, and $T_1$ has closed range, then by dividing out $K$ and using the closed graph theorem we see that the distance from $U_t f$ to $K$ approaches zero. From this fact, and the fact that $K$ is finite dimensional, it follows that a subsequence of $U_t f$ converges to a point $g$ of $K$, with $\|g\| = \|f\|$. This contradicts the weak convergence of $U_t f$ to zero.

**Corollary 4.** Let $H$ be absolutely continuous, and $A$ be $H$-compact and symmetric. Further, suppose that for some positive integer $n$, $AH^2$, $HAH$ and $H^2A$ are defined on domain $H^n$. Then the restriction of $AH - HA$ to domain $H^n$ can have no extension to a closed operator in $h$ with closed range and finite dimensional null space.

**Proof.** Combine Theorem 2 and Corollary 3.

**References**


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