SOME FIXED POINT RESULTS FOR $UV$ DECOMPOSITIONS OF COMPACT METRIC SPACES

JOHN COBB AND WILLIAM VOXMAN

Abstract. In this paper the preservation of the fixed point property under $UV$ decompositions is studied. It is shown that if $K$ is an $n$-dimensional complex with the fixed point property and $G$ is $UV^{n-1}$ decomposition of $K$, then $K/G$ also will have the fixed point property. Furthermore, if $X$ is a compact metric space with the fixed point property, and $G$ is a $UV^n$ decomposition of $X$ such that $X/G$ may be embedded in a suitably small Euclidian space, $R^n$, then $X/G$ retains the fixed point property.

Introduction. A space $X$ is said to have the fixed point property if each continuous function $f : X \to X$ leaves some point of $X$ fixed, i.e., there is a point $x \in X$ such that $f(x) = x$. In this paper it is shown that the fixed point property for compact metric spaces is preserved under suitable $UV$ decompositions. Examples and some applications of the main results are described. Proofs rely to a great extent on techniques utilized by Armentrout and Price in [1], and it would be helpful if the reader is familiar with this paper.

Notation and terminology. If $G$ is an upper semicontinuous decomposition of a topological space $X$, then $X/G$ will denote the associated decomposition space, and $P : X \to X/G$ the natural projection from $X$ onto $X/G$.

Suppose $X$ is a topological space, $M$ is a subset of $X$, and $n$ is a nonnegative integer. $M$ has property $n$-$UV$ if and only if for each open set $U$ containing $M$, there is an open set $K$ containing $M$ such that (1) $K \subseteq U$ and (2) each singular $n$-sphere in $V$ is homotopic to 0 in $U$. $M$ has property $UV^n$ if and only if for each nonnegative integer such that $i \leq n$, $M$ has property $i$-$UV$; $M$ has property $UV^n$ if and only if for each nonnegative integer $k$, $M$ has property $k$-$UV$.

If $X$ is a topological space and $n$ is a nonnegative integer, the statement that $G$ is $UV^n$ decomposition of $X$ means that $G$ is an upper semicontinuous decomposition of $X$ into compact sets, each with property $UV^n$.

Received by the editors April 15, 1971.

AMS 1969 subject classifications. Primary 54B15; Secondary 54B25.

Key words and phrases. Simplicial complex, compact metric space, $UV$ decomposition, fixed point property.
Suppose $X$ is a topological space. If $\mathcal{U}$ is a collection of subsets of $X$ and $A \subseteq X$, then the star of $A$ with respect to $\mathcal{U}$, denoted by $\text{st}(A, \mathcal{U})$, is
$$\bigcup \left\{ U : U \in \mathcal{U} \text{ and } U \text{ intersects } A \right\}.$$ Suppose $\mathcal{U}$ and $\mathcal{V}$ are collections of open subsets of $X$. Then $\mathcal{V}$ star refines $\mathcal{U}$ if and only if for each set $V$ of $\mathcal{V}$, there is a set $U$ of $\mathcal{U}$ such that $\text{st}(V, \mathcal{V}) \subseteq U$. If $n$ is any nonnegative integer, then $\mathcal{V}$ star $n$-homotopy refines $\mathcal{U}$ if and only if for each set $V$ of $\mathcal{V}$, there is a set $U$ of $\mathcal{U}$ such that (1) $\text{st}(V, \mathcal{V}) \subseteq U$ and (2) if $0 \leq k \leq n$, each singular $k$-sphere in $\text{st}(V, \mathcal{V})$ is homotopic to 0 in $U$.

**Theorem 1.** Suppose $K$ is an $n$-dimensional finite simplicial complex with the fixed point property and $G$ is a $UV^{-1}$ upper semicontinuous decomposition of $K$. Then $K/G$ has the fixed point property.

**Proof.** Suppose $h$ is a map from $K/G$ into itself. To find a fixed point for $h$, we first show that for each $\epsilon > 0$, there exists a map $F: K \to K$ such that $d(PF, hP) < \epsilon$, that is, $d(PF(x), hP(x)) < \epsilon$ for each $y$ in $K/G$ where $d$ is the metric of $K/G$. In order to apply the “lifting extension” of Price and Armentrout [1], we select an arbitrary point $x$ in $K$, and choose $x' \in P^{-1}[hP(x)]$. Let $f: \{x\} \to \{x'\}$. Since $G$ is a $UV^{-1}$ decomposition of the complex $K$, there exists by [1, Lemma 3.2, p. 435] an extension $F$ of $f$ to all of $K$ with the desired property, $d(PF, hP) < \epsilon$. Thus, for each positive integer $n$, we may find a map $F_n: K \to K$ such that $d(PF_n, hP) < 1/n$. Let $x_n$ be a fixed point of $F_n$. Passing to a subsequence if necessary we may assume that the sequence $\{x_n\}$ converges to a point $z$. Then $P(z)$ will be a fixed point of $h$ since
$$d(hP(z), P(z)) \leq d(hP(z), hP(x_n)) + d(hP(x_n), PF_n(x_n)) + d(PF_n(x_n), P(z))$$
and the sum on the right side tends towards 0 for increasing values of $n$. Since $h$ was arbitrary it follows that $K/G$ has the fixed point property.

That $G$ cannot be an arbitrary decomposition of $K$ is seen from the following easy example.

**Example 1.** Let $K$ be the unit disk and $g$ the unit circle in $K$. Then $K$ has the fixed point property while $K/G$ does not, where $K/G$ is the decomposition space obtained by identifying $g$ to a point.

A fairly natural question would be the following. Suppose $G$ is a $UV^{-\infty}$-decomposition of a finite simplicial complex $K$ such that $K/G$ has the fixed point property. Does this imply that $K$ also enjoys this property? The answer is negative as can be seen from the next example.
Example 2. Let $K$ be the space described in Fig. 1(b) of [3, p. 21]. Here $X$ is the Lopez space $\mathbb{C}P^2 \cup S_1 \times S_2 \cup \mathbb{C}P^4$ where $S_1$ and $S_2$ are 2-spheres with $S_1$ identified with $\mathbb{C}P^1 \subset \mathbb{C}P^2$ and $S_2$ identified with $\mathbb{C}P^1 \subset \mathbb{C}P^4$. $\mathbb{C}P^8$ is the suspension of complex projective 8-space. To the wedge of $X$ and $\mathbb{C}P^8$ a 2-cell is attached. As indicated in [3], $K$ does not have the fixed point property. However, if we decompose the 2-cell into straight line segments (see above drawing), then the decomposition of $K$ whose nondegenerate elements consist of precisely these segments yields a decomposition space which is the wedge of $X$ and $\mathbb{C}P^8$ and, hence, has the fixed point property [3]. This decomposition is easily seen to be $UV^0$.

If certain dimension restrictions are placed on the decomposition space, then Theorem 1 may be extended to arbitrary compact metric spaces. We first establish a lemma (Lemma 2) which is itself of some general interest.

Lemma 1 (Armentrout and Price [1]). Suppose $X$ is a metric space, $n$ is a nonnegative integer, $G$ is a $UV^n$ decomposition of $X$, and $A$ is a subset of $X/G$. If $\mathcal{U}$ is an open covering of $A$, there exists an open covering $\mathcal{V}$ of $A$ such that $\{p^{-1}[V]: V \in \mathcal{V}\}$ star $n$-homotopy refines $\{p^{-1}[U]: U \in \mathcal{U}\}$.

Lemma 2. Suppose $X$ is a metric space, $m$ and $n$ are positive integers, $G$ is a $UV^n$ decomposition of $X$, and $X/G$ is a subset of (may be embedded in) $R^m$ where $m \leq n+1$. Let $\varepsilon$ be a positive number. Then there exists an open set $U$ of $R^m$ which contains $X/G$ and a map $F: U \to X$ such that $d(PF(u), u) < \varepsilon$ for each $u \in U$.

Proof. Let $\mathcal{V}$ be a finite open covering of $X/G$ by sets of diameter less than $\varepsilon/2$. By repeated applications of Lemma 1, we may obtain a sequence, $\mathcal{V}^0, \mathcal{V}^1, \ldots, \mathcal{V}^m$ of finite open coverings of $X/G$ such that $\{p^{-1}[V]: V \in \mathcal{V}^0\}$ star $n$-homotopy refines $\{p^{-1}[W]: W \in \mathcal{V}^i\}$, and for $1 \leq i \leq m$,

$$\{p^{-1}[V_i]: V_i \in \mathcal{V}^i\}$$

star $n$-homotopy refines $\{p^{-1}[V_i-1]: V_{i-1} \in \mathcal{V}^{i-1}\}$. Let $\delta$ be a Lebesgue number for $\mathcal{V}^m$ and $T$ be a triangulation of $R^m$ with mesh less than $\min\{\delta/4, \varepsilon/2\}$. Let $N = \{\sigma \subset T: \sigma$ is an $m$-simplex and $\sigma \cap X/G \neq \emptyset\}$. For each $\sigma \in N$, choose a point $z_\sigma \in \sigma \cap X/G$, and with each vertex $v$ of a simplex in $N$, associate a point $y_v$ in $X/G$ in the following manner. If $v \in X/G$, then $y_v = v$, and if $v \notin X/G$, then $y_v$ is to be a point in $X/G$ of minimum distance from $v$. Note then that for every $\sigma \in N$, $\text{diam} \{(y_v, v$ a vertex of $\sigma) \cup \{z_\sigma\}\} < \delta$.

We now construct a function $F$ from $N^* = \bigcup \{\sigma: \sigma \in N\}$ into $X$ by first defining $F$ on the vertices, then extending $F$ to the 1-simplices, then to the 2-simplices, etc. If $v$ is a vertex of a simplex in $N$, we let $F(v) \in p^{-1}(y_v)$. Suppose $\tau$ is a 1-simplex in some simplex of $N$ with vertices $v_1$ and $v_2$. It follows from our construction that there is a $V^m \in \mathcal{V}^m$ such that $y_{v_1}$ and $y_{v_2}$ are both contained in $V^m$ (since $d(x_{v_1}, x_{v_2}) < \delta$). Hence, $F(v_1)$ and $F(v_2)$
belong to $P^{-1}[V^m]$. Since $\{P^{-1}[V^m]; V^m \in \mathcal{V}^m\}$ is a star $n$-homotopy refinement of $\{P^{-1}[V^{m-1}]; V^{m-1} \in \mathcal{V}^{m-1}\}$ there is a $V^{m-1} \in \mathcal{V}^{m-1}$ such that $F$ may be extended to a map of $\tau$ into $P^{-1}[V^{m-1}]$. Continuing in this manner (as described in [1, Lemma 3.2]) we may extend $F$ to all of $N^*$; furthermore, $F$ will have the property that for each $\sigma \in N$, there is a $V^0 \in \mathcal{V}^0$ such that $F[V^0] \subset P^{-1}[V^0]$. Observe also that $V^0$ may be chosen so that $z \in V^0$, and of course, $V^0 \subset W$ for some $W \in \mathcal{W}^*$.

Let $U = \text{Interior } N^*$. To complete the proof we need to check that if $u \in U$, then $d(PF(u), u) < \varepsilon$. For $u \in U$ there exists a $\sigma \in N$ such that $u \in \sigma$. Then it is easily verified that

$$d(PF(u), u) \leq d(PF(u), z_\sigma) + d(z_\sigma, u) < \varepsilon/2 + \varepsilon/2$$

since $PF(u)$ and $z_\sigma$ both belong to some $W \in \mathcal{W}^*$.

**Theorem 2.** Suppose $G$ is a $UV^n$ decomposition of a compact metric space $X$ such that $X/G$ may be embedded in $R^n$ for some $m \leq n + 1$. Then if $X$ has the fixed point property, so does $X/G$.

**Proof.** Suppose that $X/G$ does not have the fixed point property. Let $K: X/G \to X/G$ be a map which is fixed point free. Since $X/G$ is compact there is a positive number $\varepsilon$ such that $d(K(y), y) \geq \varepsilon$ for each $y \in X/G$. By Lemma 2, we may find an open set $U$ in $R^n$ which contains $X/G$ and a map $F: U \to X$ such that $d(PF(u), u) < \varepsilon$ for each $u \in U$. Define $g = F \circ K \circ P$. Thus $g$ is a map from $X$ into $X$ and, hence, has a fixed point $z$. But $d(KP(z), P(z)) < \varepsilon$ since $P(z) = PF(KP(z))$ and $d(PF(KP(z)), KP(z)) < \varepsilon$. Since this contradicts the fact that $d(K(y), y) \geq \varepsilon$ for each $y \in X/G$, $K$ must have a fixed point.

**Corollary 1.** Suppose $G$ is a $UV^m$ decomposition of a compact metric space $X$ such that $X/G$ is finite dimensional. Then if $X$ has the fixed point property, so does $X/G$.

**Corollary 2.** Suppose $G$ is a $UV^n$ decomposition of a compact metric space $X$ such that $\dim X/G \leq n/2$. Then if $X$ has the fixed point property, so does $X/G$.
Proof. Since \(2(n/2)+1 \leq n+1\), \(X/G\) may be embedded in \(R^{n+1}\) and Theorem 2 applies.

As an application of Theorem 2, we consider the following spaces. Let \(X\) be the cone over a circle with convergent spiral (Figure 1). Let \(Y\) be the bottomless-can-with-skirt-with-lid-attached (Figure 2; see [2, p. 131]), and let \(Z\) be the solid can-with-skirt (Figure 3).

\(X\) does not have the fixed point property (see, for example, [2, p. 129]), and, hence, \(Y\) cannot have the fixed point property; for if it did, the decomposition space obtained by identifying the lid to a point (a \(UV^\omega\) decomposition) would also have the fixed point property—but this space is just \(X\). It then follows that \(Z\) fails to have the fixed point property (see also Knill [4]) since \(Z\) retracts onto \(Y\).

Bibliography


Department of Mathematics, University of Idaho, Moscow, Idaho 83843