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SHELLING 3-CELLS IN COMPACT TRIANGULATED 3-MANIFOLDS

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ABSTRACT. Although examples of nonshellable 3-cells are known, this paper shows that every compact 3-manifold (with or without boundary) has a triangulation in which every 3-cell which is the union of 3-simplexes of the triangulation can be shelled.

Since the usefulness of shelling in piecewise linear topology is well known [**2**], [**3**], [**6**], it is of some interest to know under what circumstances an n-cell can be shelled relative to some given triangulation or cellular subdivision. In [**3**] Moise gives a proof that a 2-cell can be shelled relative to any given triangulation, and Sanderson in [**6**] shows a similar theorem relative to cellular subdivisions of 2-cells. Sanderson also shows that if the cellular subdivision has at least two elements, then there are at least two free 2-cells; and thus concludes that a given element may always be reserved until the last.

Bing's example of "the house with two rooms" shows that the Moise theorem cannot be generalized to three dimensions, but Sanderson [**6**] does give a proof that given a triangulation T of a 3-cell, then there is a subdivision of T that can be shelled. The author in [**8**] gives a bound on the number of elements in such a subdivision in terms of the number of elements in T (assuming the given 3-cell is embedded in E^3 and that the elements of T are rectilinear). In [**5**] M. E. Rudin gives an example of a rectilinear triangulation of a tetrahedron which cannot be shelled.

It is the purpose of this paper to approach the shelling problem from a somewhat different viewpoint by proving the following

THEOREM 1. *If M is a compact 3-manifold with or without boundary, then there is a triangulation T of M such that if C is a 3-cell which is the union of two or more 3-simplexes of T, then at least two 3-simplexes of T are free in C.*

In [**7**] the author proves a result similar to Theorem 1 by showing that if K is a polygonal knot in regular position in E^3, then there is a large solid tetrahedron G and a triangulation T of G such that (1) $K \subset \mathrm{Int}(G)$, (2) K

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is the union of 1-simplexes of T, and (3) if C is a 3-cell which is the union of two or more 3-simplexes of T, then at least two 3-simplexes of T are free in C. The result proved is actually stronger (and much more complicated to state).

Some definitions are now in order. The n-cell g is said to be free in the n-cell h provided $g \subset h$ and either $g=h$ or $g \cap \mathrm{Bd}(h)$ is an $n-1$ cell. If T is a cellular subdivision of the n-cell h, then h can be shelled relative to the elements of T provided there is a labeling $t_1, t_2, \cdots, t_n$ of the elements of T such that (1) if $1 \leqq p \leqq n$ then $\bigcup_{i=p}^{n} t_i$ is an n-cell in which t_p is free, and (2) if $1 \leqq i < j \leqq n$ then $t_i \neq t_j$. Such a labeling $t_1, t_2, \cdots, t_n$ is called a shelling order for h relative to the elements of T. If T is understood, then the notation may be shortened so as to not mention T. An n-manifold is a separable metric space such that each point of it has a neighborhood homeomorphic to Euclidean n-space E^n. An n-manifold with boundary is a separable metric space such that each point of it has a neighborhood homeomorphic to a closed topological n-cell. The other definitions in this paper are standard and may be found in **[1]**, **[3]**, **[6]**.

Proof of Theorem 1. By Moise **[4]** and Bing **[1]**, M can be triangulated, so let T be a triangulation of M which has no more 3-simplexes than any other triangulation of M. Assuming that T does not satisfy the conclusion of the theorem, let C be a 3-cell in M which (1) is the union of at least two 3-simplexes of T, (2) does not contain two 3-simplexes of T which are free in C, and (3) does not contain properly a 3-cell C' satisfying (1) and (2). In each of the following seven cases a contradiction will be obtained.

Case 1. There is a 3-simplex $g=abcd$ of T lying in C such that $g \cap \mathrm{Bd}(C) = abc \cup abd \cup bcd$. Since $\mathrm{Cl}(C-g)$ is a 3-cell in M which is the union of one or more 3-simplexes of T, there is a 3-simplex g' of T which is free in $\mathrm{Cl}(C-g)$ and does not have a 2-simplex common with g unless $g' = \mathrm{Cl}(C-g)$. Both g and g' are free in C, a contradiction.

Case 2. Suppose Case 1 does not hold, but there is a 2-simplex abc of T lying in C such that $abc \cap \mathrm{Bd}(C) = \mathrm{Bd}(abc)$. In this case $C-abc$ is the union of two mutually separated sets C_1 and C_2 such that $\bar{C}_i$ $(i=1, 2)$ is a 3-cell. There is a 3-simplex g_i $(i=1, 2)$ which is free in $\bar{C}_i$ and does not contain abc unless $g_i = \bar{C}_i$. Both g_1 and g_2 are free in C, a contradiction.

Case 3. Assuming none of the previous cases hold, suppose there are two 3-simplexes g and g' of T which lie in C and have two 2-simplexes common with $\mathrm{Bd}(C)$. In this case each of g and g' would intersect $\mathrm{Bd}(C)$ in precisely the union of two 2-simplexes, and would thus be free in C, a contradiction.

Case 4. Suppose none of the previous cases hold, but there is a 3-simplex $g=abcd$ of T which lies in C, where $g \cap \mathrm{Bd}(C) = abc$. Let g_1, g_2, g_3

be, respectively, the 3-simplexes of T which lie in C and intersect g in a 2-simplex. If g_i ($i=1, 2$ or 3) has a 2-simplex common with $\mathrm{Bd}(C)$, then g and g_i are both free in C, so suppose g_i ($i=1, 2, 3$) has no such 2-simplex. Let $f: C \to Q$ be a homeomorphism of C onto a solid geometric tetrahedron Q in E^3 and let $q \in \mathrm{Int}(Q)$. For each 2-simplex S of T lying on $\mathrm{Bd}(C)$ let $t(S)$ denote the union of all straight line intervals $qf(s)$ for $s \in S$, and let T' be the triangulation of M whose 3-simplexes are either 3-simplexes of T which are not subsets of C or of the form $f^{-1}(t(S))$. Since there is at most one 3-simplex of T which lies in C and has two 2-simplexes common with $\mathrm{Bd}(C)$, the triangulation T' has less 3-simplexes than T, a contradiction.

Case 5. Suppose none of the previous cases hold, but there is a 3-simplex $g=abcd$ of T lying in C, where $g \cap \mathrm{Bd}(C)=abc \cup bcd$. The conditions of this case imply that if $g'=xyzw$ is a 3-simplex of T lying in C, $g \neq g'$, and $xyz \subset g' \cap \mathrm{Bd}(C)$, then $g' \cap \mathrm{Bd}(C)=xyz \cup \{w\}$, $xyz \cup xw$, $xyz \cup yw$ or $xyz \cup zw$. See [5].

If there are two 3-simplexes of T which contain b and c, respectively, and lie except for their 0-simplexes in $\mathrm{Int}(C)$, then the proof may be finished as in Case 3 by defining a new triangulation T' of M with fewer 3-simplexes than T. Therefore suppose that every 3-simplex of T which contains b and lies in C has at least one 2-simplex common with $\mathrm{Bd}(C)$.

Let $g_1=adbe$ be the 3-simplex of T which lies in C and has adb common with g. It follows that $g_1 \cap \mathrm{Bd}(C)=edb \cup ab$ or $eab \cup bd$.

Let $t_1, t_2, \cdots, t_m$ be distinct 2-simplexes of T lying in C such that (0) $t_1=abc$ and $db \subset t_m$, (1) there is a positive integer j ($1<j \leqq m$) such that $t_j \not\subset \mathrm{Bd}(C)$, but if i is a positive integer ($1 \leqq i \leqq m$) and $i \neq j$, then $t_i \subset \mathrm{Bd}(C)$, (2) t_i and t_j are 2-simplexes of the same 3-simplex of T for $i=j-1$ or $j+1$, (3) if p is a positive integer ($1 \leqq p < m$) then $t_p \cap t_{p+1}$ is a 1-simplex of the form xb, and (4) m is maximal.

Let $t_j=xyb$ and let $t=xybu$ be the 3-simplex of T in C which contains t_j but neither t_{j-1} nor t_{j+1}. Since $t \cap \mathrm{Bd}(C)=xbu \cup yb$ or $ybu \cup xb$, it is evident that 2-simplexes $t_1', t_2', \cdots, t_{m+1}'$ may be found satisfying (0)–(3) above. This contradicts the assumption that m is maximal.

Case 6. Suppose none of the previous cases hold, but there is a 3-simplex $g=abcd$ of T such that $g \cap \mathrm{Bd}(C)=abc \cup bd$. If there is a 3-simplex of T lying in C and having no 2-simplex common with $\mathrm{Bd}(C)$, then a contradiction may be obtained as in Case 3. Let $g_1=abde$ be the 3-simplex of T having abd common with g. A contradiction may now be obtained as in Case 5.

Case 7. Suppose none of the previous cases hold. Let $g=abcd$ be a 3-simplex of T lying in C where $abc \subset \mathrm{Bd}(C)$. Then $g \cap \mathrm{Bd}(C)=\{d\} \cup abc$. Let $h_1, h_2, \cdots, h_n$ be distinct 2-simplexes of T lying in $\mathrm{Bd}(C)$ such that (1) $abc=h_1$, (2) dh_i is a 3-simplex of T lying in C, $1 \leqq i \leqq n$, (3) if $1 < p \leqq n$, then h_p has a 1-simplex common with some h_i, $1 \leq i < p$, and (4) n is maximal. If every 2-simplex of T lying in Bd(C) is an h_i ($1 \leq i \leq n$), then d is not a 0-simplex of any 2-simplex of T lying in Bd(C), a contradiction. The proof is now finished as in Case 4, and all possible cases are now exhausted.

THEOREM 2. *The conclusion of Theorem* 1 *may be changed to require that if g is a* 3*-simplex of T lying in C, then there is a shelling of C relative to the* 3*-simplexes of T lying in C, where g is saved until the last.*

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