PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

EMBEDDING PARTIALLY ORDERED SPACES IN TOPOLOGICAL SEMILATTICES[1]

LLOYD D. TUCKER

ABSTRACT. A partial order Γ on a compact space S is called continuous if Γ is a closed subset of $S\times S$. In this paper, we define and study an embedding Φ of the arbitrary compact continuously partially ordered space (S, Γ) into a corresponding compact topological semilattice S_Γ. We show that the structure of S_Γ entirely determines the structure of (S, Γ). We prove that the inverse images under Φ of components in S_Γ are the order components of (S, Γ), where elements a and b of S are defined to be in the same order component of (S, Γ) if there exists no continuous monotonic map $f:(S, \Gamma)\to\{0, 1\}$ which separates a and b. Finally, we show that S_Γ is connected if and only if (S, Γ) has only one order component.

Notation. If S is a topological space, we denote the space of nonempty closed subsets of S by 2^S. If $\{V_j | j=1, \cdots, n\}$ is a set of open subsets of S, we let $\langle V_1, \cdots, V_n\rangle=\{X\subset 2^S | X\cap V_j\neq\varnothing$ for each $j\leqq n$ and $X\subset V_1\cup \cdots \cup V_n\}$. We give 2^S the finite topology [1], a basis for which is $\{\langle V_1, \cdots, V_n\rangle | V_1, \cdots, V_n$ are open subsets of $S\}$.

A *quasi order* is a transitive reflexive binary relation. A *partial order* is an antisymmetric quasi order. If Γ is a quasi order on the Hausdorff space S and Γ is closed with respect to the product topology on $S\times S$, then Γ is a *continuous* quasi order. We write CCQOTS (CCPOTS) for "compact continuously quasi (partially) ordered topological space(s)." Let (S, Γ) and (T, Ω) be CCQOTS. If $x\in S$ and $A\subset S$, then we let $x\Gamma=\{y\in S | (x, y)\in\Gamma\}$, and $A\Gamma=\bigcup\{a\Gamma | a\in A\}$. It is easy to see [4] that if A is compact then $A\Gamma$ is closed. We say $f:(S, \Gamma)\to(T, \Omega)$ is *monotonic* if $f:S\to T$ is a function such that if $(a, b)\in\Gamma$, then $(f(a), f(b))\in\Gamma'$. If $f:(S, \Gamma)\to(T, \Omega)$ is monotonic, one-to-one, and onto, and $f^{-1}:(T, \Omega)\to(S, \Gamma)$ is monotonic, then f is an *isomorphism*. We let

$$E(\Gamma) = \{(x, y)\in S\times S \mid (x, y), (y, x)\in\Gamma\},$$
$$\Gamma/E(\Gamma) = \{(a, b)\in S/E(\Gamma)\times S/E(\Gamma) \mid \text{if } x\in a \text{ and } y\in b \text{ then } (x, y)\in\Gamma\},$$

Received by the editors April 30, 1971.

AMS 1970 *subject classifications*. Primary 54F05; Secondary 06A20.

Key words and phrases. Continuous orders, embedding, topological semilattice, order component, order connected.

[1] This research was supported in part by a grant from the National Science Foundation.

© American Mathematical Society 1972

$$U(\Gamma) = \{(A, B) \in 2^S \times 2^S \mid A\Gamma \supset B\Gamma\},$$
$$S' = 2^S/E(U(\Gamma)) \quad \text{and} \quad \Gamma' = U(\Gamma)/E(U(\Gamma)).$$

We say $(T, *)$ is a topological semilattice if $*$ is a continuous commutative associative operation on the space T such that $t*t=t$ for each $t\in T$.

PROPOSITION 1. *If* (S, Γ) *is a CCPOTS, then* $(2^S, U(\Gamma))$ *is a CCQOTS, and* (S', Γ') *is a CCPOTS.*

PROOF. Well known.

DEFINITION OF S_Γ. If (S, Γ) is a CCPOTS, let $F(\Gamma)$ be the set of all continuous monotonic $f:(S, \Gamma)\to I$, where I is the unit interval with the usual topology and order. Let $C=C(F(\Gamma))$ be the cube $\{g:F(\Gamma)\to I\}$, with the product topology and the natural partial order $\leqq$, defined by pointwise minimum. Specifically, we define $*:2^C\to C$ by $\pi_f(*(A))=\inf\{\pi_f(a)\mid a\in A\}$ where π_f is the projection map, and we say $a\leqq b$ if $*(\{a, b\})=a$. Let $\Phi:S\to C$ be the usual embedding of S into C defined by $\pi_f(\Phi(s))=f(s)$. We let S_Γ be the smallest (topologically) closed subsemilattice of C which contains $\Phi(S)$.

NOTE. If (S, Γ) is a CCPOTS, then by [**2**, p. 30], $F(\Gamma)$ separates points of S, so Φ is one-to-one. Thus Φ is a homeomorphism into C. Also, the partial order Γ agrees with $\leqq$ in the sense that $(a, b)\in\Gamma$ if and only if $\Phi(a)\leqq\Phi(b)$. Thus, we assume without loss of generality that S is a subspace of C.

PROPOSITION 2. *If* (S, Γ) *is a CCPOTS, and* $i:(2^S, U(\Gamma))\to S$ *is defined by* $i(A)=*(A)$*. then i is a continuous monotonic extension of the embedding* $\Phi:(S, \Gamma)\to S_\Gamma$ *such that i is onto* S_Γ*, and if* $i(A)\leqq i(B)$ *for some* $A, B\in 2^S$*, then* $(A, B)\in U(\Gamma)$.

PROOF. Let $T=\{*(A)\mid A\in 2^S\}$. Since $*$ is a continuous operation and 2^S is compact, T is closed. Clearly, T is a subsemilattice of C containing S, so $S_\Gamma\subset T$. Let $\mathscr{F}(S)$ be the set of all nonempty finite subsets of S. By [**1**, p. 156], $\mathscr{F}(S)$ is dense in 2^S. Since $*(A)\in S_\Gamma$ for each $A\in\mathscr{F}(S)$, we see that $T\subset S_\Gamma$. Therefore $T=S_\Gamma$. One may now show i satisfies the desired conclusion by applying the theorem of Nachbin [**2**, p. 30] which states that if $a\in S$, $B\in 2^S$, and $a\notin B\Gamma$, then there exists $f\in F(\Gamma)$ such that f separates a and B.

COROLLARY 3. *If* (S, Γ) *is a CCPOTS and* $q:2^S\to S'$ *is the quotient map, then there exists an isomorphism* $j:(S', \Gamma')\to S_\Gamma$ *such that j is a homeomorphism and the following diagram commutes.*

PROOF. Straightforward.

$$\begin{tikzcd} (2^S, U(\Gamma)) \arrow[rr, "q"] \arrow[dr, "i"'] & & (S', \Gamma') \arrow[dl, "j"] \\ & S_\Gamma & \end{tikzcd}$$

DEFINITION. If X is a topological semilattice and $A \subset X$, then A is a *minimal closed generating subset of* X if A is closed, A generates X in the sense that the only closed subsemilattice of X containing A is X, and no closed proper subset of A generates X.

PROPOSITION 4. *If (S, Γ) is a CCPOTS, then $\Phi(S)$ is the unique minimal closed generating subset of S_Γ.*

PROOF. Suppose on the contrary that there exists a closed generating subset A of S_Γ such that S is not a subset of A. Let $T=\{*(D) \mid D \in 2^A\}$. One can show $T=S_\Gamma$ by using an argument similar to the proof of Proposition 2. Let s be an element of $S\backslash A=\{x\in S \mid x\notin A\}$.

There exists $B\in 2^A$ such that $*(B)=s$. Let $P=\{E\in 2^S \mid i(E)\in B\}$. Since P is closed in 2^S, one can show $\bigcup P$ is closed in S. We want to show $\bigcup P\subset s\Gamma$. Suppose on the contrary there exists $x\in \bigcup P\backslash s\Gamma$. Then by [**2**, p. 30] there exists $f\in F(\Gamma)$ such that $f(x)=0$ and $f(t)=1$ for each $t\in s\Gamma$. Clearly, $\pi_f(i(\bigcup P))=0$ and $\pi_f(s)=1$. But $i(\bigcup P)=*(\bigcup P)=*(B)=s$. This contradiction indicates that $\bigcup P\subset s\Gamma$. Similarly, one can show $s\in \bigcup P$. Thus, there exists $E\in P$ such that $s\in E\subset s\Gamma$. Since $(E, \{s\}), (\{s\}, E)\in U(\Gamma)$ and i is monotonic, we see that $i(E)=i(\{s\})=s$. Thus $s\in B\subset A$. This contradiction indicates $S\subset A$. Therefore, S is a minimal generating closed subset of S_Γ.

COROLLARY 5. *If (S, Γ) and (T, Ω) are CCPOTS, and S_Γ and T_Ω are isomorphic and homeomorphic, then (S, Γ) and (T, Ω) are isomorphic and homeomorphic.*

PROOF. By Proposition 4, we can recover the topological and order structures of (S, Γ) and (T, Ω) from the topological and order structures of S_Γ and T_Ω respectively.

DEFINITION. In [3], the author defined the relation $C_3(\Gamma)$ on the CCPOTS (S, Γ) by

$$C_3(\Gamma) = \{(a, b) \in S \times S \mid \text{there exists } no\ f \in F(\Gamma) \text{ such that the image of } f \text{ is } \{0, 1\} \text{ and } f(a)\neq f(b)\}.$$

The equivalence classes of $C_3(\Gamma)$ are called *order components*. If $C_3(\Gamma)=S\times S$, then (S, Γ) is *order connected.*

PROPOSITION 6. *If (S, Γ) is a CCPOTS and $a, b\in S$, then a and b lie in the same order component of (S, Γ) if and only if $\Phi(a)$ and $\Phi(b)$ lie in the same component of S_Γ.*

PROOF. First, suppose $(a, b) \in S \times S \setminus C_3(\Gamma)$. There exists $f \in F(\Gamma)$ such that the image of f is $\{0, 1\}$, and $f(a) \neq f(b)$. Thus, $S_\Gamma = \pi_f^{-1}(\{0, 1\})$, and a and b lie in the disjoint open sets $\pi_f^{-1}(0)$ and $\pi_f^{-1}(1)$ respectively, so $\Phi(a) = a$ and $\Phi(b) = b$ lie in distinct components of S.

Now suppose $\Phi(a)$ and $\Phi(b)$ lie in distinct components of S_Γ. By **[3]**, the order components and the components of a compact topological semilattice are identical, so there exists $g \in F(\leqq)$ such that the image of g is $\{0, 1\}$ and $g(a) \neq g(b)$. Clearly, $g \circ \Phi \in F(\Gamma)$, the image of $g \circ \Phi$ is $\{0, 1\}$, and $g \circ \Phi(a) \neq g \circ \Phi(b)$. Thus, a and b lie in distinct order components of (S, Γ).

PROPOSITION 7. *If (S, Γ) is a CCPOTS, then S is order connected if and only if S_Γ is connected.*

PROOF. ($\Leftarrow$) Apply Proposition 6.

($\Rightarrow$) Suppose S_Γ is not connected. By **[3]**, the order components and the components of S_Γ are identical, so there exists $g \in F(\leqq)$ such that the image of g is $\{0, 1\}$. Let $\mathscr{F}(S)$ be the set of finite subsets of S. Since $(g \circ i)^{-1}(0)$ is a nonempty open subset of 2^S, and since by **[1]** $\mathscr{F}(S)$ is dense in 2^S, there exists $A \in \mathscr{F}(S) \cap (g \circ i)^{-1}(0)$. Let $B \in (g \circ i)^{-1}(1)$, and let $b \in B$. Define $F(U(\Gamma))$ just as in the case where the order is a partial order. Clearly $g \circ i \in F(U(\Gamma))$, $g \circ i(\{b\}) = 1$, and $g \circ i(A \cup \{b\}) = 0$. Let D be a subset of $A \cup \{b\}$ such that $b \in D \in (g \circ i)^{-1}(0)$, and no proper subset of D which contains b is an element of $(g \circ i)^{-1}(0)$. There exists $d \in D \setminus \{b\}$.

Define $h: S \to 2^S$ by $h(s) = \{s\} \cup (D \setminus \{d\})$. Clearly, $h: (S, \Gamma) \to (2^S, U(\Gamma))$ is continuous and monotonic, so $g \circ i \circ h \in F(\Gamma)$. But $g \circ i \circ h(d) = 0$, $g \circ i \circ h(b) = 1$, and the image of $g \circ i \circ h$ is $\{0, 1\}$. Thus (S, Γ) is not order connected.

REFERENCES

1. E. A. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR **13**, 54.

2. L. Nachbin, *Sur les espaces topologiques ordonnés*, C. R. Acad. Sci. Paris **226** (1948), 381–382; English transl. in *Topology and order*, Van Nostrand Math. Studies, Van Nostrand, Princeton, N.J., 1965. MR **9**, 367; MR **36** #2125.

3. Lloyd D. Tucker, *Generalized components and continuous orders* (to appear).

4. L. E. Ward, Jr., *Concerning Koch's theorem on the existence of arcs*, Pacific J. Math. **15** (1965), 347–355. MR **31** #6206.

DEPARTMENT OF MATHEMATICS, BOISE STATE COLLEGE, BOISE, IDAHO 83707