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ON GROUPS OF EXPONENT FOUR. III

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ABSTRACT. C. R. B. Wright has shown that the nilpotency class of an n-generator group of exponent four is at most $3n-1$. In this paper, it is shown that if this bound can be improved to $3n-3$ then the free group of exponent four of infinite rank is solvable.

Introduction. Let $\kappa(n)$ and $\lambda(n)$ denote the nilpotency class and the solvability length respectively of the free n-generator group of exponent 4. If $\mu(n)$ denotes the nilpotency class of the free n generator metabelian group of exponent 4, then $\mu(n)=n+1$ if $n\geqq 4$ and $\mu(n)=n+2$ if $n=2$, 3 (Gupta-Tobin [1]). Since $\kappa(n)\geqq\mu(n)$, $\kappa(n)\to\infty$ as $n\to\infty$. On the other hand, Marshall Hall, Jr. (verbal communication) and G. Higman [4] have conjectured that there exists an integer l such that $\lambda(n)\leqq l$ for all $n=2, 3, \cdots$. In this connection, the reader is also referred to Tritter [5] and Gupta-Weston [2]. It is a well-known result of C. R. B. Wright [6] that $\kappa(n)\leqq 3n-1$. In this paper, we establish a connection between the Hall-Higman conjecture and the exact value of $\kappa(n)$. Our main result is as follows:

THEOREM. *If $\lambda(k)\to\infty$ as $k\to\infty$, then for $n\geqq 3$, $\kappa(n)=3n-2$ or $3n-1$.*

Notation. $[x,y]=x^{-1}y^{-1}xy$; $[x,y,z]=[[x,y],z]$; $\gamma_m(G)$ is the mth term of the lower central series of G; $\langle x^G\rangle$ is the normal closure of x in G. A commutator is of type $(r\to s)$ if it is of weight s in r variables.

Preliminaries. In this section, we shall state and prove a number of lemmas required for the proof of the main result.

LEMMA 1. *Let n be a fixed integer exceeding 2 and let G be a group of exponent 4 in which every commutator of type $(n\to 3n-2)$ is trivial. For each $m\geqq 0$, let $C_m=[x_{i(1)}, y_1, \cdots, y_m, x_{i(2)}, \cdots, x_{i(3n-3)}]$ where $|\{i(1), \cdots, i(3n-3)\}|\leqq n-1$. Then $C_m\in\gamma_4(\langle x_{i(j)}^G\rangle)$ for some $j=1, 2, \cdots, 3n-3$.*

PROOF. Since $|\{i(1), \cdots, i(3n-3)\}|\leqq n-1$, we may assume that for some r and s, $1<r<s\leqq 3n-3$, $i(1)=i(r)=i(s)$. We now argue that

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$C_m \in \gamma_4(\langle x_{i(1)}^G \rangle)$. Since $[x_{i(1)}, y_1, \cdots, y_m] \in \langle x_{i(1)}^G \rangle$, we have $[x_{i(1)}, y_1, \cdots, y_m] = \prod_{j=1}^t x_{i(1)}^{\varepsilon_j z_j}$ where $\varepsilon_j \in \{-1, 1\}$, and $z_j \in G$. Thus in turn we have

$$C_m = \left[\prod_{j=1}^{t} x_{i(1)}^{\varepsilon_j z_j}, x_{i(2)}, \cdots, x_{i(3n-3)}\right]$$
$$= \prod_{j=1}^{t} [x_{i(1)}^{z_j}, x_{i(2)}, \cdots, x_{i(3n-3)}]^{\varepsilon_j} \bmod \gamma_4(\langle x_{i(1)}^G \rangle)$$

(since $i(1)=i(r)=i(s)$ for some $r, s, 1<r<s$). Further, since $x_{i(1)}^{z_j} = x_{i(1)}[x_{i(1)}, z_j]$, we have

$$[x_{i(1)}^{z_j}, x_{i(2)}, \cdots, x_{i(3n-3)}]$$
$$= [x_{i(1)}, \cdots, x_{i(3n-3)}][x_{i(1)}, z_j, x_{i(2)}, \cdots, x_{i(3n-3)}] \bmod \gamma_4(\langle x_{i(1)}^G \rangle).$$

But the first factor on the right-hand side is of type $(n-1 \to 3n-3)$ and is trivial by the result of Wright; and the second factor is of type $(n \to 3n-2)$ and is trivial by hypothesis.

LEMMA 2 (WRIGHT [6]). *Let G be a group of exponent 4 and let $n \geqq 2$, $r \geqq 1$ and $p \geqq 0$ be fixed integers. Then*

$$[u, x, y, x, x, v_1, \cdots, v_r, w_1, \cdots, w_s]$$
$$= [u, y, v_1, \cdots, v_{r-1}, x, v_r, x, x, w_1, \cdots, w_s] \bmod \gamma_{n+r+s+5}(G)$$

for all $u \in \gamma_n(G)$ and all $x, y, v_1, \cdots, v_r, w_1, \cdots, w_s \in G$ (this lemma is obtained by repeated application of congruences (8) and (9) of [6]).

LEMMA 3 (WRIGHT [6]). *Let G be a group of exponent 4 and let $i(r) = i(s) = i(t) = q$ for some integers r, s, and t with $2<r<s<t \leqq n$. Then modulo $\gamma_{n+1}(G)$, $[x_{i(1)}, \cdots, x_{i(n)}]$ can be expressed as a product of commutators of the form*

$$[y_{j(1)}, \cdots, y_{j(n-4)}, x_q, y_{j(n-3)}, x_q, x_q]$$

and

$$[y_{j(1)}, \cdots, y_{j(n-3)}, x_q, x_q, x_q]$$

where the y_j's are the x_i's in some order.

LEMMA 4 (WRIGHT [6]). *Let G be a group of exponent 4 and let $n \geqq 1$ and $m \geqq 0$ be fixed integers. Then modulo $\gamma_{n+m+2}(G)$, $[x_1, \cdots, x_n, y, z_1, \cdots, z_m]$ can be expressed as a product of commutators of the form $[y, x_{1\sigma}, \cdots, x_{n\sigma}, z_1, \cdots, z_m]$ where σ is a permutation of $\{1, \cdots, n\}$.*

LEMMA 5. *Let G be a group of exponent 4 and let C be a commutator of type $(k+m \to 3k+2m+l)$ where $k \geqq 3$, $m \geqq 0$, $l \geqq 0$. Then modulo $\gamma_{3k+2m+l+1}(G)$, C can be written as a product of commutators of the form*

$$[x_{i(1)}, y_1, \cdots, y_{2m+l+6}, x_{i(2)}, \cdots, x_{i(3k-6)}]$$

where $|\{i(1), \cdots, i(3k-6)\}|=k-2$ *and* $\{y_1, \cdots, y_{2m+l+6}\}$ *consists of the remaining* $m+2$ *variables.*

PROOF.[1] By Theorem 1 of Wright [6], we may assume that no commutator contains an entry more than 3 times. We prove the lemma by induction on $k\geqq 3$. If $k=3$, there are at least 3 entries in C which appear 3 times and one of them, say $x_{i(1)}$, is not one of the first two entries of C. Thus by Lemmas 3 and 4, C can be written as a product of commutators of the form

$$[x_{i(1)}, y_1, \cdots, y_{2m+l+6}, x_{i(1)}, x_{i(1)}],$$

where $\{y_1, \cdots, y_{2m+l+6}\}$ consists of the remaining $m+2$ variables. Now let $k>3$ and assume that the result has been proved for $k-1$. Write $3k+2m+l=3(k-1)+2(m+1)+(l+1)$. By the induction hypothesis modulo $\gamma_{3(k-1)+2(m+1)+(l+1)+1}(G)$, C can be written as a product of commutators of the form

$$[x_{i(1)}, y_1, \cdots, y_{2m+l+9}, x_{i(2)}, \cdots, x_{i(3k-9)}]$$

where $|\{i(1), \cdots, i(3k-9)\}|=k-3$ and $\{y_1, \cdots, y_{2m+l+9}\}$ consists of the remaining $(m+1)+2$ variables. Thus there are at least 3 variables which appear 3 times in $\{y_1, \cdots, y_{2m+l+9}\}$ and one of them, say $x_{i(p)}$, is different from y_1. By Lemma 3, modulo $\gamma_{2m+l+11}(G)$,

$$[x_{i(1)}, y_1, \cdots, y_{2m+l+9}]$$

is a product of commutators of the form

$$[z_1, \cdots, z_{2m+l+6}, x_{i(p)}, z_{2m+l+7}, x_{i(p)}, x_{i(p)}]$$

and $[z_1, \cdots, z_{2m+l+7}, x_{i(p)}, x_{i(p)}, x_{i(p)}]$. Thus modulo $\gamma_{3k+2m+l+1}(G)$, using Lemma 2 to the appropriate form, C can be written as a product of commutators of the form

$$[z_1, \cdots, z_{2m+l+7}, x_{j(2)}, \cdots, x_{j(3k-6)}],$$

where $|\{j(2), \cdots, j(3k-6)\}|=k-2$, $x_{i(1)}=x_{j(1)}\in\{z_1, \cdots, z_{2m+l+7}\}$ and $\{z_1, \cdots, z_{2m+l+7}\}\backslash\{x_{j(1)}\}$ consists of the remaining $m+2$ variables. A final application of Lemma 4 gives the desired result.

LEMMA 6. *Let* $n\geqq 2$ *be a fixed positive integer and let* G *be a group of exponent* 4 *such that*

(i) *every commutator of type* $(n\rightarrow 3n-2)$ *is trivial,*

(ii) $\gamma_4(\langle x^G\rangle)=\{1\}$ *for all* x *in* G.

Then for all $m\geqq 2$, *every commutator of type* $(m\rightarrow 2m+n+1)$ *is trivial.*

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[1] We thank the referee for pointing out an oversight in an earlier proof of Lemma 5.

PROOF. Let G be generated by m elements $x_1, \cdots, x_m$. Since G is nilpotent, we may assume that $\gamma_{2m+n+2}(G)=\{1\}$ and conclude that $\gamma_{2m+n+1}(G)=\{1\}$. If $m \leqq n+1$, then by the result of Wright, $\gamma_{2m+n+1}(G)=\{1\}$. Thus we may assume that $m \geqq n+2$. Let C be a typical generator of $\gamma_{2m+n+1}(G)$. Then C is a commutator of weight $2m+n+1$ with entries from $x_1, \cdots, x_m$. Since C is of type

$$((n+1)+(m-n-1) \rightarrow 3(n+1)+2(m-n-1)),$$

by Lemma 5 C can be written as a product of commutators of the form

$$[x_{i(1)}, y_1, \cdots, y_t, x_{i(2)}, \cdots, x_{i(3n-3)}]$$

where $t=2(m-n)+4 \geqq 8$, $x_{i(j)} \in \{x_1, \cdots, x_m\}$ and $|\{i(1), \cdots, i(3n-3)\}| \leqq n-1$. By Lemma 1, each such commutator is trivial. This proves the lemma.

LEMMA 7 (GUPTA-GUPTA [3]). *Let G be a group of exponent 4 and let n be a fixed positive integer such that for all $m \geqq 2$ every commutator of type $(m \rightarrow 2m+n+1)$ is trivial. Then G is solvable of bounded length.*

(This lemma is an immediate corollary of the main theorem proved in [3].)

PROOF OF THE MAIN THEOREM. Let $\lambda(k) \rightarrow \infty$ as $k \rightarrow \infty$ and let $n \geqq 3$ be the least positive integer such that $\kappa(n) \leqq 3n-3$. Let F be the free group of exponent 4 of countably infinite rank. Since $\kappa(n) \leqq 3n-3$, every commutator in F of type $(n \rightarrow 3n-2)$ is trivial. We shall prove that F is solvable of bounded length which is contrary to the hypothesis.

Let H be the group of exponent 4 generated by $a_1, a_2, \cdots$ and satisfying only the following relations and their consequences:

(I) $a_i^2=1$, $i=1, 2, \cdots$.

(II) $[a_i, h, a_i]=1$ for all a_i, $i=1, 2, \cdots$, and all h in H.

In Gupta-Weston [2], it is shown that F is solvable if and only if H is solvable. Thus we may assume that in H every commutator of type $(n \rightarrow 3n-2)$ is trivial and conclude that H is solvable. Let us assume further that $\gamma_4(\langle h^H \rangle)$ is trivial for all $h \in H$. By Lemma 6, for $m \geqq 2$, every commutator of type $(m \rightarrow 2m+n+1)$ is trivial and by Lemma 7, H is solvable of bounded length. Thus it remains for us to show that for all $h \in H$, $\gamma_4(\langle h^H \rangle)=\{1\}$. Let $K_{(i,n,h)}$, $n \geqq 1$, be the subgroup of H generated by $a_{i(1)}, \cdots, a_{i(n)}$, and where $a_{i(1)}, \cdots, a_{i(n)} \in \{a_1, a_2, \cdots\}$. By Lemma 3 of [3], $\gamma_{n+4}(K_{(i,n,h)})=\{1\}$. Since $\gamma_4(\langle h^H \rangle)$ is generated by all $\gamma_{n+4}(K_{(i,n,h)})$ for $n=1, 2, \cdots$, it follows that $\gamma_4(\langle h^H \rangle)=\{1\}$. This completes the proof of the main theorem.

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