ON GROUPS OF EXPONENT FOUR. III

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Abstract. C. R. B. Wright has shown that the nilpotency class of an $n$-generator group of exponent four is at most $3n-1$. In this paper, it is shown that if this bound can be improved to $3n-3$ then the free group of exponent four of infinite rank is solvable.

Introduction. Let $\kappa(n)$ and $\lambda(n)$ denote the nilpotency class and the solvability length respectively of the free $n$-generator group of exponent 4. If $\mu(n)$ denotes the nilpotency class of the free $n$ generator metabelian group of exponent 4, then $\mu(n)=n+1$ if $n\geq 4$ and $\mu(n)=n+2$ if $n=2, 3$ (Gupta-Tobin [1]). Since $\kappa(n)\geq \mu(n)$, $\kappa(n)\to \infty$ as $n\to \infty$. On the other hand, Marshall Hall, Jr. (verbal communication) and G. Higman [4] have conjectured that there exists an integer $l$ such that $\lambda(n)\leq l$ for all $n = 2, 3, \cdots$. In this connection, the reader is also referred to Tritter [5] and Gupta-Weston [2]. It is a well-known result of C. R. B. Wright [6] that $\kappa(n)\leq 3n-1$. In this paper, we establish a connection between the Hall-Higman conjecture and the exact value of $\kappa(n)$. Our main result is as follows:

Theorem. If $\lambda(k)\to \infty$ as $k\to \infty$, then for $n\geq 3$, $\kappa(n)=3n-2$ or $3n-1$.

Notation. $[x, y]=x^{-1}y^{-1}xy$; $[x, y, z]=[[x, y], z]$; $\gamma_m(G)$ is the $m$th term of the lower central series of $G$; $\langle x^G \rangle$ is the normal closure of $x$ in $G$. A commutator is of type $(r\to s)$ if it is of weight $s$ in $r$ variables.

Preliminaries. In this section, we shall state and prove a number of lemmas required for the proof of the main result.

Lemma 1. Let $n$ be a fixed integer exceeding 2 and let $G$ be a group of exponent 4 in which every commutator of type $(n\to 3n-2)$ is trivial. For each $m\geq 0$, let $C_m=[x_{i(1)}, y_{i(1)}, \cdots, y_{i(m)}, x_{i(2)}, \cdots, x_{i(3n-3)}]$ where $|\{i(1), \cdots, i(3n-3)\}|\leq n-1$. Then $C_m\in \gamma_4(\langle x_{i(j)}^G \rangle)$ for some $j=1, 2, \cdots, 3n-3$.

Proof. Since $|\{i(1), \cdots, i(3n-3)\}|\leq n-1$, we may assume that for some $r$ and $s$, $1<r<s\leq 3n-3$, $i(1)=i(r)=i(s)$. We now argue that
\[ C_m \in \gamma_k (\langle x_{i(1)}^G \rangle) \]. Since \([x_{i(1)}, y_1, \ldots, y_m] \in (x_{i(1)}^G),\] we have \([x_{i(1)}, y_1, \ldots, y_m] = \prod_{j=1}^{t} x_{i(1)}^{\varepsilon_j} \] where \(\varepsilon_j \in \{ -1, 1 \} \), and \(z_j \in G\). Thus in turn we have

\[ C_m = \prod_{j=1}^{t} x_{i(1)}^{\varepsilon_j}, \ldots, x_{i(3n-3)} \]

(since \(i(1) = i(r) = i(s)\) for some \(r, s, 1 < r < s\)). Further, since \(x_{i(1)}^{\varepsilon_1} = x_{i(1)}^{x_{i(1)}}, x_{i(2)}, \ldots, x_{i(3n-3)}\), we have

\[ [x_{i(1)}, x_{i(2)}, \ldots, x_{i(3n-3)}] = [x_{i(1)}, \ldots, x_{i(3n-3)}] = x_{i(1)}^{x_{i(1)}}, x_{i(2)}, \ldots, x_{i(3n-3)} \] mod \(\gamma_k (\langle x_{i(1)}^G \rangle)\).

But the first factor on the right-hand side is of type \((n-1 \rightarrow 3n-3)\) and is trivial by the result of Wright; and the second factor is of type \((n \rightarrow 3n-2)\) and is trivial by hypothesis.

**Lemma 2 (Wright [6]).** Let \(G\) be a group of exponent 4 and let \(n \geq 2, r \geq 1\) and \(p \geq 0\) be fixed integers. Then

\[ [u, x, y, x, x, v_1, \ldots, v_r, w_1, \ldots, w_s] = [u, y, v_1, \ldots, v_r, x, x, x, w_1, \ldots, w_s] \] mod \(\gamma_n+\gamma+G\)

for all \(u \in \gamma_n(G)\) and all \(x, y, v_1, \ldots, v_r, w_1, \ldots, w_s \in G\) (this lemma is obtained by repeated application of congruences (8) and (9) of [6]).

**Lemma 3 (Wright [6]).** Let \(G\) be a group of exponent 4 and let \(i(1) = i(s) = i(t)\) for some integers \(r, s, t\) with \(2 < r < s < t \leq n\). Then modulo \(\gamma_{n+1}(G), [x_{i(1)}, \ldots, x_{i(n)}]\) can be expressed as a product of commutators of the form

\[ [y_j, x_{t-1}, \ldots, x_{t-1}, y_{j+1}, x_{t-1}, \ldots, x_{t-1}] \]

and

\[ [y_j, x_{t-1}, \ldots, x_{t-1}, q, q, x_{t-1}] \]

where the \(y_i\)'s are the \(x_i\)'s in some order.

**Lemma 4 (Wright [6]).** Let \(G\) be a group of exponent 4 and let \(n \geq 1\) and \(m \geq 0\) be fixed integers. Then modulo \(\gamma_{n+m+2}(G), [x_1, \ldots, x_n, y_1, z_1, \ldots, z_m]\) can be expressed as a product of commutators of the form \([y, x_{i_1}, \ldots, x_{i_r}, x_{n+1}, \ldots, x_{n+1}, z_1, \ldots, z_m]\) where \(\alpha\) is a permutation of \(\{1, \ldots, n\}\).

**Lemma 5.** Let \(G\) be a group of exponent 4 and let \(C\) be a commutator of type \((k+m-3k+2m+l)\) where \(k \geq 3, m \geq 0, l \geq 0\). Then modulo \(\gamma_{3k+2m+1+1}(G), C\) can be written as a product of commutators of the form

\[ [x_{i(1)}, y_1, \ldots, j_{2m-1+6}, x_{i(2)}, \ldots, x_{i(3k-3)}] \]
where \([i(1), \cdots, i(3k-6)] = k-2\) and \(\{y_1, \cdots, y_{2m+1+9}\}\) consists of the remaining \(m+2\) variables.

**Proof.** By Theorem 1 of Wright [6], we may assume that no commutator contains an entry more than 3 times. We prove the lemma by induction on \(k \geq 3\). If \(k = 3\), there are at least 3 entries in \(C\) which appear 3 times and one of them, say \(x_{i(1)}\), is not one of the first two entries of \(C\). Thus by Lemmas 3 and 4, \(C\) can be written as a product of commutators of the form

\[
[x_{i(1)}, y_1, \cdots, y_{2m+1+6}, x_{i(1)}, x_{i(1)}],
\]

where \(\{y_1, \cdots, y_{2m+1+9}\}\) consists of the remaining \(m+2\) variables. Now let \(k > 3\) and assume that the result has been proved for \(k-1\). Write \(3k+2m+1 = 3(k-1)+2(m+1)+(l+1)\). By the induction hypothesis modulo \(\gamma (3k-1)+2(m+1)+(l+1)+1(G)\), \(C\) can be written as a product of commutators of the form

\[
[x_{i(1)}, y_1, \cdots, y_{2m+1+9}, x_{i(2)}, \cdots, x_{(3k-9)}],
\]

where \([i(1), \cdots, i(3k-9)] = k-2\) and \(\{y_1, \cdots, y_{2m+1+9}\}\) consists of the remaining \((m+1)+2\) variables. Thus there are at least 3 variables which appear 3 times in \(\{y_1, \cdots, y_{2m+1+9}\}\) and one of them, say \(x_{i(p)}\), is different from \(y_1\). By Lemma 3, modulo \(\gamma (2m+1+13(G))\),

\[
[x_{i(1)}, y_1, \cdots, y_{2m+1+9}]
\]

is a product of commutators of the form

\[
[z_1, \cdots, z_{2m+1+6}, x_{i(p)}, z_{2m+1+7}, x_{i(p)}, x_{i(9)}]
\]

and \([z_1, \cdots, z_{2m+1+7}, x_{i(p)}, x_{i(p)}, x_{i(9)}]\). Thus modulo \(\gamma (3k+2m+1+1(G))\), using Lemma 2 to the appropriate form, \(C\) can be written as a product of commutators of the form

\[
[z_1, \cdots, z_{2m+1+7}, x_{i(2)}, \cdots, x_{i(3k-6)}],
\]

where \([j(2), \cdots, j(3k-6)] = k-2\), \(x_{i(1)} = x_{i(1)} \in \{z_1, \cdots, z_{2m+1+7}\}\) and \(\{z_1, \cdots, z_{2m+1+7}\}\) \(\{x_{i(3)}\}\) consists of the remaining \(m+2\) variables. A final application of Lemma 4 gives the desired result.

**Lemma 6.** Let \(n \geq 2\) be a fixed positive integer and let \(G\) be a group of exponent 4 such that

(i) every commutator of type \((n \to 3n-2)\) is trivial,
(ii) \(\gamma _4(x^G) = \{1\}\) for all \(x\) in \(G\).

Then for all \(m \geq 2\), every commutator of type \((m \to 2m+n+1)\) is trivial.

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1 We thank the referee for pointing out an oversight in an earlier proof of Lemma 5.
Proof. Let $G$ be generated by $m$ elements $x_1, \ldots, x_m$. Since $G$ is nilpotent, we may assume that $\gamma_{2m+n+2}(G) = \{1\}$ and conclude that $\gamma_{2m+n+1}(G) = \{1\}$. Thus we may assume that $m \leq n+2$. Let $C$ be a typical generator of $\gamma_{2m+n+1}(G)$. Then $C$ is a commutator of weight $2m+n+1$ with entries from $x_1, \ldots, x_m$. Since $C$ is of type

$$(n+1) + (m-n-1) \rightarrow 3(n+1) + 2(m-n-1),$$

by Lemma 5 $C$ can be written as a product of commutators of the form

$$[x_{i(1)}, y_1, \ldots, y_t, x_{i(2)}, \ldots, x_{i(3n-3)}]$$

where $t = 2(m-n)+4 \geq 8$, $x_{i(j)} \in \{x_1, \ldots, x_m\}$ and $|\{i(1), \ldots, i(3n-3)\}| \leq n-1$. By Lemma 1, each such commutator is trivial. This proves the lemma.

Lemma 7 (Gupta-Gupta [3]). Let $G$ be a group of exponent 4 and let $n$ be a fixed positive integer such that for all $m \geq 2$ every commutator of type $(m-n) \rightarrow 2m+n+1$ is trivial. Then $G$ is solvable of bounded length.

(This lemma is an immediate corollary of the main theorem proved in [3].)

Proof of the Main Theorem. Let $\lambda(k) \rightarrow \infty$ as $k \rightarrow \infty$ and let $n \geq 3$ be the least positive integer such that $\kappa(n) \leq 3n-3$. Let $F$ be the free group of exponent 4 of countably infinite rank. Since $\kappa(n) \leq 3n-3$, every commutator in $F$ of type $(n \rightarrow 3n-2)$ is trivial. We shall prove that $F$ is solvable of bounded length which is contrary to the hypothesis.

Let $H$ be the group of exponent 4 generated by $a_1, a_2, \ldots$ and satisfying only the following relations and their consequences:

1. $a_i^2 = 1$, $i=1, 2, \ldots$.
2. $[a_i, h, a_j] = 1$ for all $a_i$, $i=1, 2, \ldots$, and all $h$ in $H$.

In Gupta-Weston [2], it is shown that $F$ is solvable if and only if $H$ is solvable. Thus we may assume that in $H$ every commutator of type $(n \rightarrow 3n-2)$ is trivial and conclude that $H$ is solvable. Let us assume further that $\gamma_4(h^H)$ is trivial for all $h \in H$. By Lemma 6, for $m \geq 2$, every commutator of type $(m \rightarrow 2m+n+1)$ is trivial and by Lemma 7, $H$ is solvable of bounded length. Thus it remains for us to show that for all $h \in H$, $\gamma_4(h^H) = \{1\}$. Let $K_{(i,n), h}$, $n \geq 1$, be the subgroup of $H$ generated by $a_{i(1)}, \ldots, a_{i(n)}$, and where $a_{i(1)}, \ldots, a_{i(n)} \in \{a_1, a_2, \ldots\}$. By Lemma 3 of [3], $\gamma_{n+4}(K_{(i,n), h}) = \{1\}$. Since $\gamma_4(h^H)$ is generated by all $\gamma_{n+4}(K_{(i,n), h})$ for $n=1, 2, \ldots$, it follows that $\gamma_4(h^H) = \{1\}$. This completes the proof of the main theorem.
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BIBLIOGRAPHY


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