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UNEXPECTED BEHAVIOR FOR SOLUTIONS OF A SECOND ORDER, SELFADJOINT EQUATION

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ABSTRACT. Positive, continuous functions $a(t)$ and $p(t)$ such that $a(t)p(t)\equiv t$ and for which some solution of the selfadjoint equation $(pu')'+au=0$ satisfies $\limsup_{t\to\infty}|u(t)|>0$ are shown to exist.

It is well known that all solutions of

$$(p(t)u')' + a(t)u = 0, \tag{1}$$

where $a(t)$ and $p(t)$ are positive continuous functions whose product is nondecreasing, unbounded and of class $C'[0,\infty)$, are bounded. This follows immediately from the fact that

$$\frac{d}{dt}\left\{u^2 + \frac{(pu')^2}{ap}\right\} = -\frac{(ap)'(pu')^2}{(ap)^2} \leqq 0.$$

It is also known [1], for the particular case when $p(t)\equiv 1$, that if $a(t)$ is a positive concave or convex, nondecreasing, unbounded function of class $C^2[0,\infty)$ all solutions of (1) satisfy $u(t)\to 0$ as $t\to\infty$. One might expect the same type of result to hold for solutions of the selfadjoint equation in which the product $a(t)p(t)$ is either concave or convex and is of class $C^2[0,\infty)$. It is possible, however, to construct a counterexample to this conjecture.

Let $A(s)$ be a positive, unbounded function of class $C^1[0,\infty)$ such that $A(0)=0$, $A'(s)>1$, and some solution $v_0(s)$ of

$$v''(s) + A(s)v(s) = 0 \tag{2}$$

satisfies $\limsup_{t\to\infty}|v_0(s)|>0$. D. Willett [2] has shown this existence of such a function. Now transform (2) into the form (1) by letting $t=A(s)$ and making the identification $p(t)=A'(A^{-1}(t))$, $a(t)=t/p(t)$. Clearly, $p(t)a(t)\equiv t$ satisfies the stated requirements. However, the solution $u(t)=v_0(A^{-1}(t))=v_0(s)$ of (1) in this case does not satisfy $u(t)\to 0$ as $t\to\infty$.

REFERENCES

1. A. Meir, D. Willett and J. S. W. Wong, *On the asymptotic behavior of solutions of $x''+a(t)x=0$*, Michigan Math. J. **14** (1967), 47–52. MR **35** #463.

2. D. Willett, *On an example in second order linear ordinary differential equations*, Proc. Amer. Math. Soc. **17** (1966), 1263–1266. MR **34** #1612.

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