PERTURBATIONS OF DISSIPATIVE OPERATORS WITH RELATIVE BOUND ONE

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Abstract. Let $A$ be the generator of a $(C_0)$ contraction semigroup on a Banach space. Let $B$ be a dissipative operator with densely defined adjoint. Assume that the inequality $\|Bx\| \leq \|Ax\| + b\|x\|$ holds on the domain of $A$. Then the closure of $A + B$ generates a $(C_0)$ contraction semigroup.

Let $A$ be the generator of a $(C_0)$ contraction semigroup on a Banach space $X$. Let $B$ be a dissipative operator on $X$ in the sense of Lumer and Phillips [3]. Assume that $\mathcal{D}(B) \supset \mathcal{D}(A)$. Since $A$ is closed it follows that there are constants $a, b < \infty$ such that, for every $x \in \mathcal{D}(A)$,

$$\|Bx\| \leq a\|Ax\| + b\|x\|.$$  

(1)

We say that $B$ is bounded relative to $A$ and refer to $a$ as a relative bound.

Gustafson [1], generalizing basic work of Rellich, Kato, and others (cf. [2]), showed that if the bound $a$ in (1) can be taken strictly less than 1 it follows that $A + B$ is the generator of a $(C_0)$ contraction semigroup. This is known to fail for $a > 1$. On the other hand, Wüst [4] recently showed that if $A$ and $B$ are symmetric operators on a Hilbert space with $A$ selfadjoint then the validity of (1) with $a = 1$ implies that $A + B$ is essentially selfadjoint, i.e., has selfadjoint closure. (Kato [2] had proved a slightly weaker result, starting from the analogue of (1) with norms replaced by their squares.)

In this note we use a simplified version of Wüst’s argument to extend the result to dissipative operators in a rather general Banach space setting.

Theorem. Let $X$ be a Banach space. Let $A$ and $B$ be as above with $\mathcal{D}(B) \supset \mathcal{D}(A)$. Assume that there is a constant $b < \infty$ such that, for all $x \in \mathcal{D}(A)$,

$$\|Bx\| \leq \|Ax\| + b\|x\|.$$  

(2)

Suppose also that the adjoint $B^*$ has a dense domain in $X^*$. Then the closure of $A + B$ is the generator of a $(C_0)$ semigroup.

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Remarks. 1. Our proof can easily be modified to go through under the assumption that $A^*$, rather than $B^*$, is densely defined.

2. If the space $X$ is reflexive then $A^*$ is densely defined. This uses only the fact that $A$ is closed and densely defined; the familiar "graph" argument, given by von Neumann in Hilbert space, works perfectly well in any reflexive Banach space.

Thus the hypothesis on $B^*$ is not needed if $X$ is reflexive.

3. The result of Wüst follows from the proof of our theorem.

Proof of the theorem. First note that $A+B$ is dissipative and therefore closable with dissipative closure [3, Lemma 3.3]. By the Hille-Yosida-Phillips characterization of semigroup generators it is enough to show that $I-(A+B)$ has dense range.

Suppose that $y^* \in X^*$ annihilates the range of $I-(A+B)$. We can find $y \in X$ with $\|y\| = \|y^*\|$ and $\langle y^*, y \rangle \geq \frac{1}{2} \|y^*\|^2$.

Now, by the theorem of Gustafson quoted earlier, it follows from (2) that $A+tB$ is a semigroup generator for $0 \leq t < 1$. Hence for each such $t$ there is an element $x_t \in \mathcal{D}(A+tB) = \mathcal{D}(A)$ with

(3) \quad y = (1 - A - tB)x_t.

We have $\|x_t\| \leq \|y\|$ since $A+tB$ is dissipative.

Furthermore, by (2),

$$
\|Bx_t\| \leq \|Ax_t\| + b \|x_t\| \\
\leq \|(A + tB)x_t\| + t \|Bx_t\| + b \|x_t\| \\
= \|x_t - y\| + t \|Bx_t\| + b \|x_t\|
$$

whence

$$
\|(1 - t)Bx_t\| \leq \|x_t - y\| + b \|x_t\| \leq (1 + b) \|x_t\| + \|y\|,
$$

that is,

(4) \quad \|(1 - t)Bx_t\| \leq (2 + b) \|y\|.

Now $\{x_t\}$ is bounded, and therefore it has a subnet $x_{t'}$ which converges in the weak * sense as $t' \to 1$ to some element $x^{**} \in X^{**}$. We shall show that $(1-t')Bx_{t'}$ converges weakly to 0.

For this, suppose $z^* \in \mathcal{D}(B^*)$. Then we have

(5) \quad \langle z^*, (1 - t')Bx_{t'} \rangle = (1 - t')\langle B^* z^*, x_{t'} \rangle \\
\to 0 \cdot \langle B^* z^*, x^{**} \rangle = 0.

But since $\mathcal{D}(B^*)$ is dense in $X^*$ and uniform boundedness holds by (4), a standard approximation argument shows that (5) is valid for all $z^* \in Y^*$. In particular it holds for $y^*$. 
But
\[ \langle y^*, y \rangle = \langle y^*, (1 - A - tB)x \rangle \]
\[ = \langle y^*, (1 - A - B)x_t + (1 - t)Bx_t \rangle = \langle y^*, (1 - t)Bx_t \rangle \]
since \( y^* \) kills \((1 - A - B)x_t \) by assumption.

Hence
\[ \frac{1}{t} \| y^* \|^2 \leq \langle y^*, y \rangle = \lim_{t \to 1} \langle y^*, (1 - t')Bx \rangle = 0. \]

This shows that \( y^* = 0 \). Hence the range of \( 1 - A - B \) is dense. \( \square \)

*Note added in proof.* On November 24, 1971, I received from Professor
N. Okazawa a preprint entitled “A perturbation theorem for linear con-
traction semigroups on reflective Banach spaces”. His main result is the
same as ours (for the case of reflexive spaces).

**References**

4. R. Wüst, *Generalisations of Rellich’s theorem on perturbation of (essentially) self-

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