PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 1, May 1972

A NOTE ON THE ASYMPTOTIC BEHAVIOR OF AN INTEGRAL EQUATION

CAROL SHILEPSKY[1]

ABSTRACT. Assume the existence and boundedness of a solution to an integral equation. Conditions are found which ensure the solution has a limit at infinity.

1. **Introduction.** Consider the integral equation

$$x'(t) + \int_{-\infty}^{\infty} g(x(t-\xi))\, dA(\xi) = f(t) \tag{1.1}$$

where g, A and f are prescribed real functions, and x is a bounded solution of (1.1) on $(-\infty, \infty)$. We use NBV to denote normalized bounded variation, LAC for absolutely continuous on any compact interval of $(-\infty, \infty)$, $V(A, [t_1, t_2])$, the total variation of A on $[t_1, t_2]$, and $V(A) = V(A, (-\infty, \infty))$. Consider the following hypotheses.

$H(f)$: $f \in L^\infty(-\infty, \infty)$, $\lim_{t\to\infty} f(t) = f(\infty)$ exists.

$H(g)$: $g \in C(-\infty, \infty)$, $S = \{c \mid g(c)A(\infty) = f(\infty)\}$ is nonempty and contains no interval.

$H'(g)$: $g \in C(-\infty, \infty)$, S contains exactly one point.

$H(A)$: $A(t) = A_1(t) + A_2(t)$, $A_1(t) = 0$ $(-\infty < t \leqq 0)$,

$$A_1(t) = \rho_1 > 0 \ (0 < t < \infty), \qquad A_2(t) \in \mathrm{NBV}(-\infty, \infty),$$

and $V(A_2) = \rho_2 < \rho_1$.

THEOREM 1. *Let $H(f)$, $H(g)$ and $H(A)$ hold. Let $x(t) \in \mathrm{LAC}(-\infty, \infty) \cap L^\infty(-\infty, \infty)$, let $x'(t)$ exist for each t and let $x(t)$ satisfy* (1.1) *on* $(-\infty, \infty)$. *Then*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty} x'(t) = 0$$

for some $c \in S$.

Received by the editors June 11, 1971 and, in revised form, July 23, 1971.

AMS 1970 *subject classifications*. Primary 45M05, 45G99; Secondary 45D05.

Key words and phrases. Integral equation, stability theory, Volterra equation, asymptotic behavior, bounded solution.

[1] This research was supported by the U.S. Army Research Office, Durham.

THEOREM 2. *Let $H(f)$, $H'(g)$ and $H(A)$ hold. Let $x(t)\in \mathrm{LAC}(-\infty, \infty)\cap L^\infty(-\infty, \infty)$ satisfy* (1.1) *a.e. on* $(-\infty, \infty)$. *Then*

$$\lim_{t\to\infty} x(t) = c, \qquad \lim_{t\to\infty}\left[\operatorname*{ess\,sup}_{t\leqq\tau<\infty} |x'(\tau)|\right] = 0$$

for $c\in S$.

Theorem 1 was obtained by Levin and Shea **[2]** under the additional hypothesis that g is either strictly increasing or strictly decreasing. Theorem 2 assumes that x is a solution of (1.1) only a.e. on $(-\infty, \infty)$ and was obtained independently by Levin and Shea and by the author. A proof appears in **[1]**.

It is shown in **[1]** that the requirement $\rho_2<\rho_1$ is necessary in $H(A)$. It is obvious that the requirement, x a bounded solution, is necessary in Theorems 1 and 2. For if $g(x)=-x$, $A(t)=0$ $(-\infty<t\leqq 0)$, $A(t)=1$ $(0<t<\infty)$, $f(t)\equiv 0$, then $c=0$ and (1.1) reduces to $x'(t)-x(t)=0$ which has $x(t)=e^t$ as a solution.

For further discussion of (1.1) and references to applications in ordinary differential equations and Volterra equations see **[1]** and **[2]**.

2. **Proof of Theorem 1.** We first note that $g(c)$ is the same for all c in S. Let $x(t)\in\mathrm{LAC}(-\infty, \infty)\cap L^\infty(-\infty, \infty)$ satisfy (1.1). Let

$$\alpha = \sup_{-\infty<t<\infty} |g(x(t)) - g(c)|$$

where $c\in S$, $\bar\alpha=\limsup_{t\to\infty} |g(x(t))-g(c)|$. Suppose $\bar\alpha>0$. Let $\varepsilon>0$, and, using $\rho_2<\rho_1$, let T_ε be chosen such that

$$(\rho_2 - \rho_1)\bar\alpha + (\rho_2 + \rho_1 + \alpha + 1)\varepsilon < -d$$

for some positive constant d,

$$\begin{gathered}\sup_{t>T_\varepsilon}|g(x(t)) - g(c)| < \bar\alpha + \varepsilon,\\ V(A_2, [T_\varepsilon, \infty)) < \varepsilon, \quad \text{and}\\ \sup_{t>T_\varepsilon}|f(t) - f(\infty)| < \varepsilon.\end{gathered} \tag{2.1}$$

By definition of $\bar\alpha$, there exist $\{t_n\}_{n=1}^\infty$ and $\{\varepsilon_n\}_{n=1}^\infty$ with $t_1>2T_\varepsilon$, $\varepsilon_1<\varepsilon$ such that $t_n\to\infty$ $(n\to\infty)$, $\varepsilon_n\downarrow 0$ $(n\to\infty)$ and either

$$\bar\alpha - \varepsilon_n < g(x(t_n)) - g(c) < \bar\alpha + \varepsilon_n, \quad \text{or} \tag{2.2}$$

$$-\bar\alpha - \varepsilon_n < g(x(t_n)) - g(c) < -\bar\alpha + \varepsilon_n \qquad (n = 1, 2, \cdots). \tag{2.3}$$

Suppose (2.2) holds. From (1.1) and $H(A)$ we have

$$x'(t) + \rho_1 g(x(t)) = -\int_{-\infty}^{\infty} g(x(t - \xi))\, dA_2(\xi) + f(t)$$

and using $f(\infty)=g(c)A(\infty)=g(c)(\rho_1+A_2(\infty))$,

$$x'(t)+\rho_1(g(x(t))-g(c)) = -\int_{-\infty}^{\infty}(g(x(t-\xi))-g(c))\,dA_2(\xi)+f(t)-f(\infty).$$

Let $t=t_n$. By (2.1) and (2.2) we have

$$\begin{aligned} x'(t_n)+\rho_1(\bar\alpha-\varepsilon_n) &\leqq \left\{-\int_{-\infty}^{T_\varepsilon}-\int_{T_\varepsilon}^{\infty}\right\}(g(x(t_n-\xi))-g(c))\,dA_2(\xi)+f(t_n)-f(\infty) \\ &\leqq (\bar\alpha+\varepsilon)\rho_2+\alpha\varepsilon+\varepsilon, \end{aligned}$$

and hence by (2.1),

$$x'(t_n) < -d. \tag{2.4}$$

By a similar argument, $t>t_n$ and $x(t)=x(t_n)$ imply $x'(t)<-d$. Hence $t>t_n$ and x continuous imply $x(t)<x(t_n)$. In particular,

$$-\|x\|_\infty \leqq x(t_{n+1}) < x(t_n).$$

Thus there exists x^* such that $x(t_n)\downarrow x^*$. By (2.2), $g(x^*)-g(c)=\bar\alpha$. Since $g\in C(-\infty,\infty)$, there exists $x^{**}>x^*$ such that, for $x^*<x(t)<x^{**}$, $\bar\alpha-\varepsilon<g(x(t))-g(c)<\bar\alpha+\varepsilon$. Choose N such that $x^*<x(t_N)<x^{**}$ and hence for $t>t_N$, $x^*<x(t)<x^{**}$. Then as in the proof of (2.4), we have $t>t_N$ implies $x'(t)<-d$. This contradicts $x\in L^\infty$, hence (2.2) is impossible.

Similarly (2.3) is impossible, $\bar\alpha=0$ and $\lim_{t\to\infty} g(x(t))=g(c)$, where $g(c)=g^*$ is independent of $c\in S$. Then $\lim_{t\to\infty} g(x(t))=g^*$ implies $x(t)$ approaches a value of x such that $g(x)=g^*$. That is, $\lim_{t\to\infty} x(t)=c$ for some $c\in S$. From (1.1) we then have $\lim_{t\to\infty} x'(t)=0$.

Bibliography

1. J. J. Levin and D. F. Shea, *Asymptotic behavior of the bounded solutions of some functional equations*, Contributions to Nonlinear Functional Analysis, U.S. Army Math. Res. Center, Madison, Wis., 1971.

2. ———, *On the asymptotic behavior of the bounded solutions of some integral equations*, J. Math. Anal. Appl. (to appear).

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

Current address: Box 2102, State University, Arkansas 72467