HOMOMORPHISMS OF RINGS OF GERMS OF ANALYTIC FUNCTIONS

WILLIAM R. ZAME

Abstract. Let S and S' be complex analytic manifolds with S Stein. Let X⊂S and X'⊂S' be compact sets with X holomorphically convex. Denote by \( \mathcal{O}(X) \) (respectively \( \mathcal{O}(X') \)) the ring of germs on \( X \) (respectively \( X' \)) of functions analytic near \( X \) (respectively \( X' \)). It is shown that each nonzero homomorphism of \( \mathcal{O}(X) \) into \( \mathcal{O}(X') \) is given by composition with an analytic map defined in a neighborhood of \( X' \) and taking values in \( S \).

If \( S \) and \( S' \) are complex analytic manifolds, then every analytic mapping of \( S \) into \( S' \) induces (via composition) a homomorphism of the ring of analytic functions on \( S' \) into the ring of analytic functions on \( S \). It is an important and deep result that the converse is also true, providing that \( S \) is a Stein manifold (see [1]). In this note we obtain an analogous result for homomorphisms of rings of germs of analytic functions on compact subsets of a complex analytic manifold. We show that each such homomorphism is given by composition with an analytic mapping.

1. Preliminaries and notation. Let \( S \) be a Stein manifold and \( U \) an open subset of \( S \). We denote by \( \mathcal{O}(U) \) the ring of analytic functions on \( U \). It is well known (see [1], for example) that each nonzero complex-valued homomorphism of \( \mathcal{O}(U) \) is continuous with respect to the topology on \( \mathcal{O}(U) \) of uniform convergence on compact subsets of \( U \). We denote the space of such homomorphisms by \( \mathcal{A}(\mathcal{O}(U)) \). If \( f \in \mathcal{O}(U) \) then \( f^\wedge \) denotes the function on \( \mathcal{A}(\mathcal{O}(U)) \) defined by \( \alpha \mapsto \alpha(f) \) for each \( \alpha \in \mathcal{A}(\mathcal{O}(U)) \).

Since \( S \) is Stein, \( \Delta \mathcal{O}(S) = S \), so we have the natural restriction map \( \pi_U : \Delta \mathcal{O}(U) \to S \) given by \( \pi_U(\alpha)(f) = \alpha(f|U) \). Rossi [4] has shown that \( \Delta \mathcal{O}(U) \) admits the structure of a Stein manifold in such a way that: (i) the evaluation map \( U \to \Delta \mathcal{O}(U) \) is a biholomorphism of \( U \) with an open subset of \( \Delta \mathcal{O}(U) \) (we will regard \( U \) as an open subset of \( \Delta \mathcal{O}(U) \)); (ii) if \( \alpha \in \mathcal{A}(\mathcal{O}(U)) \) then \( \alpha^\wedge \) is the unique analytic extension of \( \alpha \) to \( \Delta \mathcal{O}(U) \) (so that \( \mathcal{O}(\Delta \mathcal{O}(U)) = \mathcal{O}(U) \)); (iii) \( \pi_U \) is locally a biholomorphism.

Received by the editors September 2, 1971.


Key words and phrases. Germs of analytic functions, holomorphically convex sets.

1 Supported by National Science Foundation Grant GP-19011.
If $X$ is a compact subset of $S$, we denote by $\mathcal{O}(X)$ the ring of germs on $X$ of functions analytic near $X$. If $f$ is analytic in a neighborhood of $X$, we denote its germ on $X$ by $f$; $f$ is called a representative of $f$. We will also regard a germ in $\mathcal{O}(X)$ as a continuous function on $X$. It was shown in [2] and [6] that each nonzero complex-valued homomorphism on $\mathcal{O}(X)$ is continuous, relative to the natural inductive limit topology on $\mathcal{O}(X)$. We say that $X$ is holomorphically convex if each such homomorphism is given by evaluation at a point of $X$. Equivalently, $X$ is holomorphically convex if and only if $\{\pi_U(\Delta \mathcal{O}(U)); U$ is open and $X \subseteq U\}$ is a fundamental system of neighborhoods of $X$. More information about holomorphically convex sets may be found in [2] and [5]. We refer to [1] for general information about several complex variables.

2. Main result. Now let $S$ and $S'$ be complex analytic manifolds with $S$ Stein, and let $X \subseteq S$ and $X' \subseteq S'$ be compact sets with $X$ holomorphically convex. We wish to study homomorphisms of $\mathcal{O}(X)$ into $\mathcal{O}(X')$. If $F$ is an analytic map of a neighborhood $U'$ of $X'$ into $S$ such that $F(X') \subseteq X$, then $F$ induces a homomorphism $\Phi_F: \mathcal{O}(X) \rightarrow \mathcal{O}(X')$ as follows. Let $f$ be in $\mathcal{O}(X)$. Choose an open set $U$ containing $X$ and a representative $f$ of $f$ which is analytic on $U$. Then $f \cdot F$ is analytic on a neighborhood of $X'$; we let $\Phi_F(f)$ be the germ of $f \cdot F$ on $X'$. By a straightforward calculation, we may verify that $\Phi_F$ is a well-defined homomorphism of $\mathcal{O}(X)$ into $\mathcal{O}(X')$. Our main result is that every homomorphism arises in this way.

**Theorem.** Let $\Phi: \mathcal{O}(X) \rightarrow \mathcal{O}(X')$ be a nonzero homomorphism. Then there is an open set $U'$ containing $X'$ and an analytic function $F: U' \rightarrow S$ such that $F(X') \subseteq X$ and $\Phi = \Phi_F$. The germ of $F$ on $X'$ is unique.

For the proof of this theorem we shall make use of two lemmas. If $T$ is a subset of $S$ and $\mathcal{F}$ is a subset of $\mathcal{O}(S)$, we say that $\mathcal{F}$ separates points on $T$ if for each $x$ and $y$ in $T$ with $x \neq y$ there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. If the (complex) dimension of $S$ is $n$, we say that $\mathcal{F}$ provides local coordinates on $T$ if for each $x$ in $T$ there are $n$ functions $f_1, \cdots, f_n$ in $\mathcal{F}$ such that $d(f_1, \cdots, d(f_n(x)) \neq 0$.

**Lemma 1.** Let $\{f_1, \cdots, f_k\}$ be a subset of $\mathcal{O}(S)$ which separates points and provides local coordinates on $X$, and let $U$ be an open set containing $X$. Then there is an open set $W$ containing $X$ such that for each $x$ in $X$, each integer $M$ and each $f$ in $\mathcal{O}(U)$ vanishing to total order at least $M$ at $x$, there are functions $g_1, \cdots, g_X$ in $\mathcal{O}(W)$ and monomials $h_1, \cdots, h_N$ of degree $M$ in $f_1 - f_1(x), \cdots, f_k - f_k(x)$ such that $f = \sum g_i h_i$ in $W$.

**Proof.** A straightforward compactness argument shows that there is an open set $U_1$ containing $X$ such that $\{f_1, \cdots, f_k\}$ separates points and provides local coordinates on $U_1$. Since $X$ is holomorphically convex, there is
an open set $W$ containing $X$ such that $\pi_W(\Delta\mathcal{O}(W)) = U \cap U_1$. Let $x$ be a point of $X$, $M$ an integer, and $h_1, \cdots, h_N$ the monomials of degree $M$ in $f_1 - f_1(x), \cdots, f_k - f_k(x)$.

Let $\mathcal{O}$ be the sheaf of germs of analytic functions on $\Delta\mathcal{O}(W)$ and $\mathcal{O}^N$ the $N$-fold Cartesian product. Let $\mathcal{I}$ be the ideal sheaf of the discrete variety $\pi_W^{-1}(x)$ and $\mathcal{J}_M$ the sheaf of ideals generated by $M$-fold products of elements of $\mathcal{I}$. For each $i$, let $\phi_i = (h_i|W)^\wedge$ and let $\mu: \mathcal{O}^N \to \mathcal{J}_M$ be given by $\mu(\gamma_1, \cdots, \gamma_N) = \sum \phi_i \gamma_i$. Observe that the functions

$$((f_1 - f_1(x)|W)^\wedge, \cdots, (f_k - f_k(x)|W)^\wedge)$$

provide local coordinates on $\Delta\mathcal{O}(W)$ and vanish simultaneously only on $\pi_W^{-1}(x)$. Thus at each point of $\pi_W^{-1}(x)$, each germ in $\mathcal{J}_M$ can be expressed locally as a power series in these functions, while at each point not in $\pi_W^{-1}(x)$ at least one of these functions is locally a unit. It follows that $\mu$ is surjective.

This gives rise to the following short exact sequence of sheaves over $\Delta\mathcal{O}(W)$:

$$0 \to \ker \mu \to \mathcal{O}^N \to \mathcal{J}_M \to 0.$$  

This induces a long exact cohomology sequence, the relevant terms of which are:

$$H^0(\Delta\mathcal{O}(W), \mathcal{J}_M) \to H^0(\Delta\mathcal{O}(W), \mathcal{J}_M) \to H^1(\Delta\mathcal{O}(W), \ker \mu).$$

Since $\Delta\mathcal{O}(W)$ is a Stein manifold and $\ker \mu$ is a sheaf of relations on $\Delta\mathcal{O}(W)$, it follows from Cartan's Theorem B that $H^1(\Delta\mathcal{O}(W), \ker \mu) = 0$, so that $\mu^*$ is surjective.

Finally, if $f \in \mathcal{O}(U)$ vanishes to total order at least $M$ at $x$, then $(f|W)^\wedge = f \circ \pi_W$ vanishes to total order at least $M$ at each point of $\pi_W^{-1}(x)$. Thus $(f|W)^\wedge \in H^0(\Delta\mathcal{O}(W), \mathcal{J}_M)$. Thus $(f|W)^\wedge$ is the image under $\mu^*$ of an $N$-tuple of functions in $\mathcal{O}(\Delta\mathcal{O}(W)) = \mathcal{O}(W)$, which is the desired result.

**Lemma 2.** Let $\{f_1, \cdots, f_k\}$ be a subset of $\mathcal{O}(X)$ which separates points and provides local coordinates on $X$. If $\Phi_1$ and $\Phi_2$ are nonzero homomorphisms of $\mathcal{O}(X)$ into $\mathcal{O}(X')$ such that $\Phi_1(f_i) = \Phi_2(f_i)$ for each $i$, then $\Phi_1 = \Phi_2$.

**Proof.** Since every nontrivial homomorphism of $\mathcal{O}(X)$ into $C$ is given by evaluation at a point of $X$, we can define maps $\Phi_1^*: X' \to X$ by requiring that for each $f \in \mathcal{O}(X)$ and $y \in X'$, $\Phi_1^*(y)(f) = \Phi_1(f)(y)$ for $j = 1, 2$. For each $i$ and each $y \in X'$ we have:

$$f_i(\Phi_1^*(y)) = \Phi_1(f_i)(y) = \Phi_2(f_i)(y) = f_i(\Phi_2^*(y)).$$

Since $\{f_1, \cdots, f_k\}$ separates points on $X$, it follows that $\Phi_1^* = \Phi_2^*$.
Let \( f \) be in \( \mathcal{C}(X) \). In order to show that \( \Phi_1(f) = \Phi_2(f) \) it suffices to show that \( \Phi_1(f) - \Phi_2(f) \) vanishes to arbitrarily high order at each point of \( X' \). To this end, let \( M \) be an integer and \( y \) a point of \( X' \); set \( x = \Phi_1^*(y) = \Phi_2^*(y) \). In a neighborhood of \( x \), some representative of \( f \) may be represented by a power series in \( f_1 - f_1(x), \ldots, f_k - f_k(x) \); let \( P_M \) denote the sum of the terms of this series whose total order in \( f_1 - f_1(x), \ldots, f_k - f_k(x) \) does not exceed \( M - 1 \). Hence the germ \( f - P_M \) vanishes to total order at least \( M \) at \( x \). In view of Lemma 1, we may find germs \( g_1, \ldots, g_N \) in \( \mathcal{O}(X) \) and monomials \( h_1, \ldots, h_N \) of order \( M \) in \( f_1 - f_1(x), \ldots, f_k - f_k(x) \) such that \( f - P_M = \sum g_i h_i \).

Since \( \Phi_1 \) and \( \Phi_2 \) are nonzero, it follows that \( \Phi_1(1) = \Phi_2(1) = 1 \), so that \( \Phi_1(P_M) = \Phi_2(P_M) \). Hence

\[
\Phi_1(f) - \Phi_2(f) = \Phi_1(f - P_M) - \Phi_2(f - P_M) = \sum \Phi_1(h_i)(\Phi_1(g_i) - \Phi_2(g_i))
\]

since \( \Phi_1(h_i) = \Phi_2(h_i) \). Since \( h_i \) is the product of \( M \) germs that vanish at \( x \), \( \Phi_1(h_i) \) is the product of \( M \) germs that vanish at \( y \). It follows that \( \Phi_1(f) - \Phi_2(f) \) vanishes to order at least \( M \) at \( y \). This completes the proof.

**Proof of the Theorem.** By the imbedding theorem for Stein manifolds, we can find functions \( g_1, \ldots, g_k \) in \( \mathcal{C}(S) \) such that \( G = (g_1, \ldots, g_k) \) is a regular proper imbedding of \( S \) as a closed submanifold of \( \mathbb{C}^k \). We can find an open set \( \Omega \) containing \( G(S) \) and a holomorphic retraction \( h: \Omega \to G(S) \). If \( f \) is analytic in an open set containing \( G(X) \), then \( f \circ G \) is analytic in an open set containing \( X' \), so we have defined a homomorphism \( \nu: \mathcal{C}(G(X)) \to \mathcal{C}(X) \). Observe that \( \nu(f) = \nu(g) \) whenever \( f \) and \( g \) have representatives that agree in a \( G(S) \)-neighborhood of \( G(X) \). Let \( \Phi' = \Phi \circ \nu \) and set \( \Phi_j = \Phi'(z_j) = \Phi(g_j) \).

We can find an open set \( U' \) containing \( X' \) and functions \( q_1, \ldots, q_k \) in \( \mathcal{C}(U') \) which represent \( \Phi_1, \ldots, \Phi_k \). Set \( q = (q_1, \ldots, q_k): U' \to \mathbb{C}^k \) and consider the homomorphism \( \Phi_q: \mathcal{C}(G(X)) \to \mathcal{C}(X') \). It is easy to see that \( \Phi_q(z_i) = \Phi_j \) for each \( i \). Since \( X \) is holomorphically convex in \( S \), \( G \) is an imbedding and \( G(S) \) is closed in \( \mathbb{C}^k \), it follows that \( G(X') \) is homomorphically convex in \( \mathbb{C}^k \) (see [5]). From Lemma 2 we see that \( \Phi_q = \Phi' \).

Now set \( F = G^{-1} \circ h \circ q: U' \to S \). If \( f_i \) denotes the germ of \( g_i \circ G^{-1} \circ h \) on \( G(X) \), then we see that \( \Phi'(f_i) = \Phi'(z_i) = \Phi(g_i) \). Hence \( \Phi_F(g_i) = \Phi'(f_i) = \Phi(g_i) \). Using Lemma 2 again, it follows that \( \Phi = \Phi_F \). (Observe that this argument shows that \( q(U') \subseteq G(S) \) so that \( F = G^{-1} \circ h \circ q = G^{-1} \circ q \).) It is easy to see that \( F|X' = \Phi^* \) so that \( F(X') \subseteq X \). Finally, a straightforward calculation shows that \( \Phi_F \) depends only on the germ of \( F \) on \( X' \) and that functions representing different germs induce distinct homomorphisms.
3. Remarks. It is tempting to try to generalize the theorem to the case in which \( X \) is not holomorphically convex by passing to the space of non-zero homomorphisms of \( \mathcal{O}(X) \) into \( \mathbb{C} \). In general, however, this space admits no natural imbedding into a Stein manifold (see [2] and [6]).

If \( S' \) is Stein and \( X' \) is holomorphically convex, then \( \Phi = \Phi_F \) is an isomorphism if and only if \( F \) is a biholomorphism of a neighborhood of \( X' \) with a neighborhood of \( X \).

The author does not know whether the theorem remains true in the context of analytic spaces, rather than manifolds.

References