NOTE ON THE PROJECTIVE LIMIT ON SMALL CATEGORIES
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Abstract. Let $X$ be a small category and let $\text{Ab}^X$ be the category of covariant functors on $X$ with values in $\text{Ab}$. Consider the projective limit functor $\text{proj lim}_X : \text{Ab}^X \to \text{Ab}$. The categories $X$ for which $\text{proj lim}_X$ is exact are characterized, proving a conjecture of Oberst.

In [Bull. Amer. Math. Soc. 74 (1968), 1129–1132], U. Oberst formulated a conjecture on the exactness of the projective limit functor on the category of functors on a small category with values in the category of abelian groups.

In this note we give a proof of his conjecture. Some of the lemmas seem to have been proved by U. Oberst and J. R. Isbell by other methods.

Theorem. If $X$ is a small connected category and $\text{Ab}$ is the category of abelian groups, then the two following conditions are equivalent:

(i) For all $F \in \text{ob} \text{Ab}^X$, $\text{proj lim}_i F = 0$, for all $i \geq 1$.

(ii) There exists $y \in \text{ob} X$ such that

1. For all $x$ there exists $\xi \in X(y, x)$;
2. Every diagram

\[
\begin{array}{ccc}
 x & \xrightarrow{u} & x' \\
 \downarrow \quad & & \downarrow \\
 y & \xrightarrow{u} & y'
\end{array}
\]
in $X$ can be completed to a commutative diagram

\[
\begin{array}{ccc}
 x & \xrightarrow{u} & x' \\
 \downarrow \quad & & \downarrow \\
 y & \xrightarrow{u} & y'
\end{array}
\]
in $X$;

3. There exists $e \in X(y, y)$ such that, for all $\xi \in X(y, y)$, $\xi e = e$.

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307
Proof. Since (ii) implies that

\[ \text{proj lim } F = H^0(X(y, y), F(y)) = \{ \alpha \in F(y) \mid F(e)(\alpha) = \alpha \} \]

is trivial to see that (ii) \( \Rightarrow \) (i).

To prove that (i) implies (ii), let \( F \) be the object of \( \text{Ab}^X \) defined by

\[ F(x) = \bigoplus_{\xi \in \text{ob}(X(y))} Z \xi \text{ with } Z \xi = Z \text{ for all } \xi \in \text{ob}(X(y)). \]

Consider the obvious epimorphism \( \rho : F \rightarrow Z \) with \( Z \) the constant object of \( \text{Ab}^X \). Since proj lim is exact we have that \( \rho^*: \text{proj lim } F \rightarrow \text{proj lim } Z = Z \) is an epimorphism. Therefore there exists \( \alpha \in \text{proj lim } F \) with \( \rho^*(\alpha) = 1 \).

If \( \pi_z: \text{proj lim } F \rightarrow F(x) \) is the canonical homomorphism, then for all \( x \in X \), \( \alpha_z = \pi_z(\alpha) \in F(x) \) is nonzero. Now

\[ \alpha_z = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha^i_j(y_i) \xi^i_j, \quad \text{where } \xi^i_j \in X(y_i, x), \alpha^i_j(y_i) \in Z, \text{ and } \sum_{i=1}^{n} \alpha^i_j(y_i) = 1. \]

For at least one \( i \) we must have \( \sum_{j=1}^{m} \alpha^i_j(y_i) \neq 0 \) and we may assume that \( \alpha^i_j(y_i) \neq 0 \) for \( 1 \leq j \leq m' \leq m \).

If the diagram

\[ \begin{array}{ccc}
X & \xymatrix{ x' \ar[rr]^-\psi \ar[rr] \ar[ll]^-\varphi & & x'}
\end{array} \]

is in \( X \), then \( \varphi^* \alpha_z = \alpha_u = \psi^* \alpha_z \) and therefore

\[ \sum_j \alpha^i_j = \sum_j \alpha^i_u = \sum_j \alpha^i_z = \alpha^i_z(y_i). \]

Since \( X \) is connected it follows that \( \alpha^i_z(y_i) \neq 0 \) for all \( x \in X \) and there exist \( \xi^i_{ij} \in X(y_i, u) \) with the corresponding \( \alpha^i_u(y_i) \neq 0 \). Consequently there exist \( \xi^i_{ij} \in X(y_i, x), \xi^i_{ij} \in X(y_i, x') \) with \( \varphi \xi^i_{ij} = \xi^i_{ij} = \psi \xi^i_{ij} \), i.e. the above diagram can be completed to

\[ \begin{array}{ccc}
& y_i & \\
\begin{array}{ccc}
\xi^i_{ij} & \rightarrow & \xi^i_{ij} \\
\varphi & \rightarrow & \psi \\
X & \xymatrix{ x' \ar[rr]^-\psi \ar[rr] \ar[ll]^-\varphi & & x'}
\end{array}
\end{array} \]

We have proved (ii)(2) and at the same time (ii)(1). We need only prove
Let $F_1$ be the object of $\mathbb{A}^b_\mathbb{X}$ defined by

$$F_1(x) = \bigoplus_{\xi \in \mathcal{X}(y, x)} \mathbb{Z} \xi$$

with $y = y_i$ (i.e. the $y_i$ picked above).

By (ii)(1) there exists an epimorphism $F_1 \to Z$ in $\mathbb{A}^b_\mathbb{X}$. Since, by assumption, $Z$ is projective as an object of $\mathbb{A}^b_\mathbb{X}$, $Z$ is a direct summand of $F_1$, therefore $Z$ is a direct summand of $F_1(y)$, as an $\mathcal{X}(y, y)$-module. But $F_1(y)$ can be identified with the monoid algebra $\mathbb{Z}[\mathcal{X}(y, y)]$ and it therefore follows that the cohomology of the monoid $M = \mathcal{X}(y, y)$ is trivial.

**Lemma A.** If a monoid $M$ is cohomologically trivial, then there exists an $e \in M$ such that, for all $\xi \in M$, $\xi e = e$.

**Proof.** Consider the epimorphism $\mathbb{Z}[M] \to Z$. Since cohomology is trivial, the corresponding homomorphism $H^0(M, \mathbb{Z}(M)) \to H^0(M, Z) = Z$ is an epimorphism. Now $H^0(M, \mathbb{Z}(M)) = \{\sum_{i=1}^n \alpha_i \xi_i | \alpha_i \in \mathbb{Z}, \xi_i \in M \text{ such that, for all } \xi \in M \text{ and all } i, \xi \xi_i = \xi_j \text{ for some } j = j_i \text{ with } \alpha_i = \alpha_{j_i}\}.$

It follows that there exists an element $\sum_{i=1}^n \alpha_i \xi_i \in Z(M)$ with $\sum_{i=1}^n \alpha_i = 1$ such that for all $\xi \in M$, $\xi \xi_i = \xi_{\sigma_i(i)}$ where $\sigma_i$ is a permutation of $\{1, 2, \ldots, n\}$.

Let $S(n)$ be the symmetric group, then the correspondence $\xi \to \sigma_\xi$ gives a homomorphism $\sigma : M \to S(n)$ since $\xi' \xi \xi_i = \xi' \xi_{\sigma_i(i)} = \xi_{\sigma_{\xi'}(\sigma_i(i))}$.

Let $H = \text{im } \sigma$ then $H$ is a subgroup of $S(n)$. This follows from the fact that by the theorem of Lagrange, any submonoid of a finite group is a subgroup.

**Sublemma B.** If $M \twoheadrightarrow H$ is a surjective homomorphism of monoids and if $M$ is cohomologically trivial, then $H$ is cohomologically trivial.

**Proof.** This follows from $H^0(M, -) \to H^0(H, -)$ in the category of $H$-modules. Q.E.D.

**Sublemma C.** If a group $G$ is cohomologically trivial, then $G = \{1\}$.

**Proof.** As above there exists an element $\sum_{i=1}^n \alpha_i \xi_i \in \mathbb{Z}(G)$ such that for all $\xi \in G$, $\xi \xi_i = \xi_{\sigma_i(i)}$ and $\alpha_i = \alpha_{\sigma_i(i)}$, $\sum_{i=1}^n \alpha_i = 1$. Since for all $i, j$ there exists an element $\xi$ with $\sigma_i(j) = j$, we have $\alpha_i = \alpha_j$ for all $i, j$ and $G = \{\xi_1, \xi_2, \ldots, \xi_n\}$. Since

$$1 = n \cdot \alpha_1 = |G| \cdot \alpha_1$$

it follows that $|G| = 1$ and $\alpha_1 = 1$, and consequently $G = \{1\}$. Q.E.D.

Combining B and C we find that $\sigma_\xi = 1$ for all $\xi \in M$. This of course means that for all $\xi \in M$, $\xi \xi_i = \xi_i$. Put $\varepsilon = \xi_i$ for some $i$, and we have proved A. Q.E.D.

This ends the proof of the Theorem since A implies (ii)(3). Q.E.D.

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