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SINGULARITIES OF A CLASS OF MEROMORPHIC FUNCTIONS

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ABSTRACT. Estimates are obtained for the number of singular points, which are not poles, lying on the unit circle of the complex plane of a class of meromorphic functions which are represented by C-fractions.

I. Introduction. Representations of functions in terms of continued fractions. Given two sequences $\{a_n\}$ and $\{b_n\}$ of functions of a complex variable z, the *continued fraction*

$$b_0(z) + \operatorname*{K}_{n=1}^{\infty}\left(\frac{a_n(z)}{b_n(z)}\right) \tag{1}$$

is defined to be the sequence $\{A_n(z)/B_n(z)\}$, where the nth *approximant*

$$\frac{A_n(z)}{B_n(z)} = b_0(z) + \frac{a_1(z)}{b_1(z)} + \frac{a_2(z)}{b_2(z)} + \cdots + \frac{a_n(z)}{b_n(z)}.$$

If $A_{-1}=1$, $A_0=b_0$, $B_{-1}=0$, and $B_0=1$ and proper choices are made for arbitrary multiplicative constants, then the polynomials A_n and B_n satisfy the second order linear difference equations

$$\begin{aligned} A_n(z) &= b_n(z)A_{n-1}(z) + a_n(z)A_{n-2}(z),\\ B_n(z) &= b_n(z)B_{n-1}(z) + a_n(z)B_{n-2}(z). \end{aligned} \tag{2}$$

Various representations of functions by classes of continued fractions have been investigated. Most important are those methods which provide for each formal power series expansion

$$1 + \sum_{n=1}^{\infty} c_n z^n \tag{3}$$

exactly one continued fraction which corresponds to the series. A continued fraction (1) is said to *correspond* to a power series (3) if the power series expansion of $A_n(z)/B_n(z)$ agrees with (3) up to and including the term $c_{\nu(n)}z^{\nu(n)}$, where $\nu(n)\to\infty$ as $n\to\infty$.

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The most recent are the P-fractions introduced by Magnus in [7] and further developed by him in [8], [9] and [10]. They are of interest in particular because of their close relation to Padé tables. A continued fraction (1) is a *P-fraction* if all $a_n(z)=1$ and the $b_n(z)$ are all polynomials in z^{-1}.

The T-fractions were introduced by Thron [13] in 1948. These continued fractions are of the form $a_n(z)=z$ and $b_n(z)=1+d_nz$. They are of interest, among other reasons, because in them the elements $a_n(z)$ and $b_n(z)$ are linear functions of z. This enables one to establish fairly general convergence criteria for T-fractions. Recent work on T-fractions includes results of Waadeland [16], [17], [18] on convergence of T-fraction expansions of certain functions holomorphic in circular discs, as well as articles by Jones and Thron [5], Jefferson [4] and Hag [3]. Jones and Thron give, among other results, some theorems on the location of singular points of functions represented by T-fractions all of whose elements d_n are positive.

In 1939, Leighton and Scott [6] introduced the *C-fraction*, defined by $b_n(z)=1$ and $a_n(z)=d_nz^{\alpha_n}$, where the d_n are nonzero complex constants and the α_n are positive integers. Moduli of singular points of regular C-fractions were investigated by the present authors [1]. A central problem in the area of C-fractions is the conjecture of Leighton that meromorphic functions represented by them have the unit circle as a natural boundary, provided that $\alpha_n\to\infty$, and that the coefficients d_n are suitably restricted, for example by

$$\lim_{n\to\infty}(4\,|d_n|)^{1/\alpha_n}=1, \tag{4}$$

which would insure convergence of the C-fraction to a meromorphic function for $|z|<1$. This conjecture led to a series of investigations by Scott and Wall [11], Thron [14], [15], Singh and Thron [12] and Callas and Thron [2] which give successively better approximations to the conjecture of Leighton.

In the present paper we give substantial improvement over the previous best estimate [2] concerning the number of singular points lying on the unit circle of a class of meromorphic functions represented by C-fractions.

II. **Singularities of meromorphic functions defined on the unit disc.** Since we shall consider only C-fractions in the sequel, then in all cases $b_n(z)=1$.

If (4) holds, then it follows from the Worpitzky criterion [19, p. 42] that the C-fraction converges to a meromorphic function f for all z on the open unit disc.

Let $A_n^{(m)*}(z)/B_n^{(m)*}(z)$ be the nth approximant of the C-fraction (1), with $a_n(z)=|d_{m+n-1}|z^{\alpha_{m+n-1}}$. Let $\sigma_n^{(m)}$ and $\tau_n^{(m)}$ be the degrees of the polynomials $A_n^{(m)}(z)$ and $B_n^{(m)}(z)$, respectively, where $A_n^{(m)}(z)/B_n^{(m)}(z)$ is the nth approximant of the C-fraction (1), with $a_n(z)=d_{m+n-1}z^{\alpha_{m+n-1}}$. Let $\rho_n^{(m)}$ be the maximum of the degrees of $A_n^{(m)*}(z)$ and $B_n^{(m)*}(z)$. Then $\sigma_n^{(m)} \leqq \rho_n^{(m)}$ and $\tau_n^{(m)} \leqq \rho_n^{(m)}$.

Further, let

$$h_n = h_n^{(1)}, \qquad h_n^{(m)} = \sum_{\nu=m}^{n+1} \alpha_\nu; \tag{5}$$

$$k = k^{(1)}, \qquad k^{(m)} = \liminf_{\nu\to\infty} k_\nu^{(m)}, \qquad k_\nu^{(m)} = \frac{\rho_{n_\nu}^{(m)}}{h_{n_\nu}^{(m)} - \rho_{n_\nu}^{(m)}}; \tag{6}$$

and

$$N = N^{(1)}, \qquad N^{(m)} = \limsup_{\nu\to\infty} N_\nu^{(m)}, \qquad N_\nu^{(m)} = \frac{n_\nu - m + 2}{h_{n_\nu}^{(m)} - \rho_{n_\nu}^{(m)}}. \tag{7}$$

Using these definitions, the main result of this paper is stated as follows:

THEOREM 1. *For the C-fraction* (1), *let* $a_n(z)=d_n z^{\alpha_n}$, *where* $d_n\neq 0$ *are all complex numbers and* α_n *are all positive integers. Suppose that there exists a sequence of positive integers* $\{n_\nu\}$ *such that* $N=0$. *Let condition* (4) *be satisfied and suppose* $k>0$, *defined by* (6), *is finite. Then the meromorphic function f to which* (1) *converges on the unit disc has at least* $[(1+k)/k]+1$ *singular points, which are not poles, on the unit circle. Further, f cannot be meromorphic on any arc of the unit circle of angular measure greater than* $2\pi k/(1+k)$ *radians.*

REMARKS. Several sufficient conditions for the existence of sequences $\{n_\nu\}$ which insure that $N=0$ are given in [2].

In comparing the above result with the previous best result, consider the parameter k such that $k<10^{-2s}$, where s is the largest such positive integer. An estimate of a lower bound for the number of singular points given in [2] is then $S_1=10^s$. However the result produced by Theorem 1 is the estimate $S_2=10^{2s}$. Thus the present result gives the improvement $S_2=S_1^2$.

PROOF OF THEOREM 1. We start by stating two lemmas which are proved in [2] and [12], respectively:

LEMMA 2. *Let the power series* $P(w)=\sum_{\nu=0}^{\infty} p_\nu w^\nu$ *have radius of convergence greater than* $1+\varepsilon$, *where* $\varepsilon>0$. *Define polynomials*

$$B(z) = \sum_{\nu=0}^{N} b_\nu z^\nu \quad \textit{and} \quad B^*(z) = \sum_{\nu=0}^{N} |b_\nu|\, z^\nu.$$

Let

$$g(w) = P(w)B(\phi(w)) = \sum_{n=0}^{\infty} c_n w^n,$$

where $z=\phi(w)$ is holomorphic in a domain containing the closed disc $|w| \leqq 1+\varepsilon$, and such that

$$\max_{0 \leqq \theta < 2\pi} |\phi((1+\varepsilon)e^{i\theta})| = \phi(1+\varepsilon) > 0.$$

Then

$$|c_n| \leqq \frac{MB^*(\phi(1+\varepsilon))}{(1+\varepsilon)^n}, \quad \textit{where } M = \max_{|w|=1+\varepsilon} |P(w)|.$$

LEMMA 3. *If $\{n_\nu\}$ is a sequence such that $N=0$, then $N^{(m)}=0$ and $k^{(m)}=k$ for all m.*

Before presenting four more lemmas to be used in the proof of Theorem 1, we shall define the auxiliary function φ as follows: Let $R>1$ and $\varepsilon>0$. Then $\varphi(w, \rho)=w\{1+D(w)\rho\}$, where ρ is real, $|w|\leqq 1+\varepsilon<R$, and

$$D(re^{i\theta}) = u_1(r, \theta) + iu_2(r, \theta),$$

with

$$u_\nu(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} U(\zeta)K_\nu(r, \theta; R, \zeta)\, d\zeta$$

for $\nu=1, 2$, where

$$K_1(r, \theta; R, \zeta) = \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta - \zeta)},$$

$$K_2(r, \theta; R, \zeta) = \frac{2rR\sin(\theta - \zeta)}{R^2 + r^2 - 2rR\cos(\theta - \zeta)},$$

and

$$\begin{aligned} U(\zeta) &= U(0) > 0, && 0 \leqq \zeta \leqq \zeta_1, \\ &= U(0)(\zeta - \zeta_0)/(\zeta_1 - \zeta_0), && \zeta_1 \leqq \zeta \leqq \zeta_0, \\ &= -(\zeta - \zeta_0)/(\zeta_2 - \zeta_0), && \zeta_0 \leqq \zeta \leqq \zeta_2, \\ &= -1, && \zeta_2 \leqq \zeta \leqq \pi, \end{aligned}$$

where $U(\zeta)=U(2\pi-\zeta)$ and $U(\zeta)=U(\zeta+2\pi)$ for all ζ. Further, we require that the values $U(0)$ and ζ_0 satisfy the relations

$$(8) \quad \zeta_0 U(0) = \pi - \zeta_0 \quad \text{and} \quad kU(0) < 1 - \delta, \qquad \text{where } 0 < \delta < U(0).$$

It is easy to see that $D(\bar{w})=(D(w))^-$. Further, φ is holomorphic in w on the disc $|w|\leqq 1+\varepsilon<R$ for any ρ. Note that $\varphi(w, 0)=w$. For notational convenience, in the sequel, we shall write $u_1=u$ and $u_2=v$.

LEMMA 4. *Let $\delta>0$. Then, for sufficiently small $R>1$,*

$$|u(1+\varepsilon, \theta) - U(\theta)| < \delta$$

for all θ such that $0\leqq\theta\leqq 2\pi$ and for any ε such that $1\leqq 1+\varepsilon<R$.

PROOF. This lemma follows easily from the continuity of U on the compact set $0 \leqq \theta \leqq 2\pi$, which implies that the Poisson integral $u(r, \theta)$ satisfies $\lim_{r \to R} u(r, \theta) = U(\theta)$ uniformly with respect to $0 \leqq \theta \leqq 2\pi$.

As consequences of this lemma, it is easy to see that $D(1) = U(0)$ and $D(-1) = -1$, where $D(1) = u(1, 0)$ and $D(-1) = u(1, \pi)$.

LEMMA 5. *For all sufficiently small* $\rho > 0$,

$$\max_{0 \leqq \theta < 2\pi} |\varphi((1+\varepsilon)e^{i\theta}, \rho)| = \varphi(1+\varepsilon, \rho)$$

for all ε *such that* $1 \leqq 1+\varepsilon \leqq R_1 < R$.

PROOF. For the moment let us assume, for all θ such that $0 \leqq \theta < 2\pi$, that

$$\operatorname{Re}\{D((1+\varepsilon)e^{i\theta})\} \leqq D(1+\varepsilon), \tag{9}$$

with equality only at $\theta = 0$. Then, for all ε such that $1+\varepsilon \leqq R_1 < R$ and all sufficiently small ρ, we can easily prove that

$$2\operatorname{Re}\{D((1+\varepsilon)e^{i\theta})\} + |D((1+\varepsilon)e^{i\theta})|^2\rho \leqq 2D(1+\varepsilon) + D(1+\varepsilon)^2\rho \tag{10}$$

for all θ on $\sigma \leqq \theta \leqq 2\pi - \sigma$, where $\sigma > 0$. From the identity

$$|\varphi((1+\varepsilon)e^{i\theta}, \rho)|^2 = (1+\varepsilon)^2\{1 + 2\operatorname{Re}\{D((1+\varepsilon)e^{i\theta})\}\rho + |D((1+\varepsilon)e^{i\theta})|^2\rho^2\}$$

it is clear that if we show that (10) holds on the whole interval $0 \leqq \theta < 2\pi$, then we will have proved the lemma, provided, of course, that we prove (9).

To show that (10) holds on the interval $0 \leqq \theta < 2\pi$, we consider this inequality in terms of u and v. Since we have assumed for the moment that (9) holds, with equality only at $\theta = 0$, then

$$2 \geqq \rho\left\{\frac{v(1+\varepsilon, \theta)^2}{u(1+\varepsilon, 0) - u(1+\varepsilon, \theta)} - u(1+\varepsilon, \theta) - u(1+\varepsilon, 0)\right\} \tag{11}$$

for $\theta \neq 0$. By Taylor expansion of $u(1+\varepsilon, \theta)$ and $v(1+\varepsilon, \theta)$ about $\theta = 0$, the first term in the brackets of (11) becomes

$$\frac{2v_\theta(1+\varepsilon, 0)^2}{-u_{\theta\theta}(1+\varepsilon, 0)} + O(\theta),$$

provided $u_{\theta\theta}(1+\varepsilon, 0) \neq 0$. If $u_{\theta\theta}(1+\varepsilon, 0) \neq 0$, then ρ can be chosen so small that the inequality (11) holds, and hence (10) holds for all θ such that $0 \leqq \theta < 2\pi$. The inequality $u_{\theta\theta}(1+\varepsilon, 0) \neq 0$ can be established by first proving the Riemann-Stieltjes integral representation

$$u_{\theta\theta}(1+\varepsilon, \theta) = \frac{1}{2\pi}\int_0^{2\pi} K_1(1+\varepsilon, \theta; R, \zeta)\, dU'(\zeta).$$

Then, for $\theta=0$, it follows easily that $K(\zeta_1)>K(\zeta_0)>K(\zeta_2)$, where $K(\zeta)=K_1(1+\varepsilon, 0; R, \zeta)$, implies that $u_{\theta\theta}(1+\varepsilon, 0)<0$.

To complete the proof of the lemma it suffices to show that (9) holds with equality only at $\theta=0$. This can be done by proving that $u(1+\varepsilon, \theta)$ is strictly decreasing on $0<\theta<\pi$ and strictly increasing on $\pi<\theta<2\pi$, and that $u_\theta(1+\varepsilon, 0)=u_\theta(1+\varepsilon, \pi)=0$. This completes the proof of the lemma.

LEMMA 6. *Let $1\leqq 1+\varepsilon<R$ and $u^*\in(\delta-1, 0)$, where $0<\delta<\min\{1, U(0)\}$ and $R=R(\delta)$ is as in Lemma* 4. *Then there exists a unique $\theta^*\in(\zeta_1, \zeta_2)$ such that $u(1+\varepsilon, \theta^*)=u^*$.*

PROOF. Lemma 4 implies that $u(1+\varepsilon, \zeta_2)<\delta-1$ and $u(1+\varepsilon, \zeta_1)>U(0)-\delta>0$. Further, $u^*\in(\delta-1, 0)$ implies that there exists a θ^* such that $u(1+\varepsilon, \theta^*)=u^*$. Uniqueness follows from the fact stated in Lemma 5 that $u(1+\varepsilon, \theta)$ is strictly decreasing on the open interval $(0, \pi)$. Moreover, this monotonicity property of u implies that $\zeta_2>\theta^*>\zeta_1$. This concludes the proof of the lemma.

LEMMA 7. *Let f be the meromorphic function to which the C-fraction* (1) *converges. Define F by the composition $F(w, \rho)=f(\varphi(w, \rho))$, where $\varphi(w, \rho)$, as defined before, is holomorphic in w with ρ sufficiently small and $R>1$ such that $R-1$ is sufficiently small. Let $\Delta=\varphi(1+\varepsilon, \rho)/(1+\varepsilon)$. Assume that $F(w, \rho)$ is meromorphic for $|w|<1+\varepsilon^*$ and let $0<\varepsilon<\varepsilon^*$. Then $1+\varepsilon\leqq\Delta^k$, where k is defined by* (6).

PROOF. The meromorphic function f may be written as $f(z)=p(z)/q(z)$, where p and q are holomorphic wherever f is meromorphic and $p(0)=q(0)=1$. It is well known **[12]** that

$$p(z)B_n(z) - q(z)A_n(z) = (-1)^n\left(\prod_{\nu=1}^{n+1} d_\nu\right)z^{h_n} + \cdots, \tag{12}$$

where the omitted terms on the right are of degree higher than h_n in z. Let $\pi_n=\prod_{\nu=1}^{n+1} d_\nu$, where the product is taken only over those factors for which $|d_\nu|>1$. Let $A'_n(z)=A_n(z)/\pi_n$ and $B'_n(z)=B_n(z)/\pi_n$. Then the coefficients of the polynomials $A'_n(z)$ and $B'_n(z)$ satisfy **[12]**

$$|a_\nu^{(n)}| \leqq 2^n \quad\text{and}\quad |b_\nu^{(n)}| \leqq 2^n \tag{13}$$

for $\nu\leqq\sigma_n$ and $\nu\leqq\tau_n$, respectively. Define the compositions $P(w, \rho)=p(\varphi(w, \rho))$ and $Q(w, \rho)=q(\varphi(w, \rho))$. Then $P(0, \rho)=Q(0, \rho)=1$, and P and Q are holomorphic in w wherever F is meromorphic. The identity (12) becomes, upon letting $z=\varphi(w, \rho)$,

$$\begin{aligned} P(w, \rho)B'_n(\varphi(w, \rho)) - Q(w, \rho)A'_n(\varphi(w, \rho)) \\ = (-1)^n\left(\prod_{\nu=1;\,|d_\nu|\leqq 1}^{n+1} d_\nu\right)w^{h_n} + \cdots, \end{aligned} \tag{14}$$

where the omitted terms are of higher degree than h_n in w. Let P and Q have power series expansions

$$P(w, \rho) = 1 + \sum_{\nu=1}^{\infty} p_\nu w^\nu \quad \text{and} \quad Q(w, \rho) = 1 + \sum_{\nu=1}^{\infty} q_\nu w^\nu.$$

Then both these series have radii of convergence greater than $1+\varepsilon$. By Lemmas 2 and 5, if

$$P(w, \rho)B_n'(\varphi(w, \rho)) = \sum_{\nu=0}^{\infty} c_\nu^{(n)} w^\nu$$

and

$$Q(w, \rho)A_n'(\varphi(w, \rho)) = \sum_{\nu=0}^{\infty} e_\nu^{(n)} w^\nu,$$

then

$$(15) \quad |c_\nu^{(n)}| \leqq \frac{M_1 B_n''(\varphi(1+\varepsilon, \rho))}{(1+\varepsilon)^\nu} \quad \text{and} \quad |e_\nu^{(n)}| \leqq \frac{M_2 A_n''(\varphi(1+\varepsilon, \rho))}{(1+\varepsilon)^\nu},$$

for $\nu=0, 1, 2, \cdots$, where

$$M_1 = \max_{|w|=1+\varepsilon} |P(w, \rho)| \quad \text{and} \quad M_2 = \max_{|w|=1+\varepsilon} |Q(w, \rho)|$$

and $A_n''(z)$ and $B_n''(z)$ are the polynomials obtained from $A_n'(z)$ and $B_n'(z)$, respectively, where each coefficient is replaced by its modulus. Let $M=\max\{M_1, M_2\}$. Then it follows from the identity (14), by using the estimates (13) and (15), that

$$\prod_{\nu=1;|d_\nu|\leqq 1}^{n+1} |d_\nu| \leqq \frac{2^{n+1}M}{(1+\varepsilon)^{h_n}} \frac{\{(1+\varepsilon)\Delta\}^{\rho_n+1}}{(1+\varepsilon)\Delta - 1}.$$

In a similar manner for the C-fraction (1), with $a_n(z)=d_{m+n-1}z^{\alpha_{m+n-1}}$, where the d_ν satisfy (4), then, for fixed arbitrary η such that $0<\eta<1$ and sufficiently large m depending on η,

$$(16) \quad \frac{(1-\eta)^{h_n^{(m)}}}{4^{n-m+2}} \leqq \prod_{\nu=m;|d_\nu|\leqq 1}^{n+1} |d_\nu| \leqq \frac{2^{n-m+2}M'}{(1+\varepsilon)^{h_n^{(m)}}} \frac{\{(1+\varepsilon)\Delta\}^{\rho_n^{(m)}+1}}{(1+\varepsilon)\Delta - 1}.$$

Let $\{n_\nu\}$ be the sequence in Lemma 3, then $k=k^{(m)}$ and $N^{(m)}=0$ for all m. Taking the $(h_{n_\nu}^{(m)}-\rho_{n_\nu}^{(m)})$th roots of the extreme members of the inequalities (16) and letting $\nu\to\infty$, we get $(1-\eta)^{1+k}\leqq\Delta^k/(1+\varepsilon)$. Since $\eta>0$ is arbitrary, the lemma is proved.

The proof of Theorem 1 now shall be completed in two steps. First, an arc Γ on the unit circle will be found which is not free of singular points of the function f. Second, we shall calculate the angular measure of the arc Γ to be $2\pi k/(k+1)$ radians. Then the theorem follows easily by a rotation argument which proves that no arc of that angular measure is free of singular points.

Let $\varepsilon>0$ be the largest value for which $F(w, \rho)$ is meromorphic on the open disc $|w|<1+\varepsilon$, then $1+\varepsilon\leqq\{1+D(1+\varepsilon)\rho\}^k$. Hence $k^{-1}\varepsilon+T(\varepsilon)\leqq D(1+\varepsilon)\rho$, where $T(0)=0$ and $T(\varepsilon)/\varepsilon\to 0$ as $\varepsilon\to 0$. Define $\rho=X(\varepsilon)=(k^{-1}\varepsilon+T(\varepsilon))/D(1+\varepsilon)$. Then $X'(0)=1/kD(1)>0$. Let $\rho^*=X^*(\varepsilon)$ such that X^* is a twice continuously differentiable function, $X'(0)=X^{*\prime}(0)$, and $X(\varepsilon)>X^*(\varepsilon)$ in a neighborhood of $\varepsilon=0$. Then, for all $\varepsilon>0$ in this neighborhood,

$$1 + \varepsilon = \{1 + D(1 + \varepsilon)\rho\}^k > \{1 + D(1 + \varepsilon)\rho^*\}^k.$$

Hence, for any sufficiently small $\varepsilon>0$, $F(w, \rho^*)$ cannot be meromorphic on the open disc $|w|<1+\varepsilon$. Since $\varphi(w, \rho^*)$ is holomorphic on $|w|<1+\varepsilon$, then the singular points of $F(w, \rho^*)$ in the region $|w|\leqq 1+\varepsilon$ are caused by the singular points of the function f in the image region $\varphi(G, \rho^*)$, where $G=\{w:|w|\leqq 1+\varepsilon\}$. Since f has no singular points, other than poles, on the unit disc V, then its singularities lie in the crescent-shaped region $\varphi(G, \rho^*)-V$, assuming that there is just one intersection point of the image of the boundary of G with the unit circle in the upper half plane. As $\varepsilon\to 0$, this crescent-shaped region contracts to an arc Γ on the unit circle, which is symmetric about the point $z=1$. Therefore, there is at least one singular point of f on the arc Γ.

In calculating the angular measure of this arc Γ, for sufficiently small $\varepsilon>0$, let $\theta(\varepsilon)$ satisfy the equation

$$|\varphi((1 + \varepsilon)e^{i\theta(\varepsilon)}, X^*(\varepsilon))| = 1, \tag{17}$$

where $\theta(\varepsilon)>0$ is taken as the smallest such solution. That such a solution exists follows from the intermediate value theorem and the inequalities

$$-1 < \varphi(-(1 + \varepsilon), X^*(\varepsilon)) < 1 < \varphi(1 + \varepsilon, X^*(\varepsilon)). \tag{18}$$

Conditions which imply that (18) hold are the following: The equality $D(-1)=-1$ implies that the left-most inequality of (18) holds. Further, since we shall require that $1-kU(0)>0$ be arbitrarily small in the sequel, then the middle inequality of (18) will be satisfied also. The right-most inequality of (18) holds without conditions beyond the requirement that $k>0$.

By implicit differentiation of the squared members of the defining relation (17) of $\theta(\varepsilon)$ with respect to ε, we get, upon letting $\varepsilon\to 0$, the necessary condition $u(1, \theta_0)=-kD(1)$. Let $u^*=-kD(1)<0$. Since $D(1)=U(0)$, then the condition $-kU(0)>\delta-1$ (see (8)) implies that $u^*>\delta-1$. Hence θ_0 exists, and, by Lemma 6, θ_0 is unique and satisfies $\zeta_1<\theta_0<\zeta_2$. Therefore, by a continuity argument, for sufficiently small $\varepsilon>0$, (17) has a unique solution $\theta(\varepsilon)$.

Let $\lambda=1-kU(0)$. Then $0<\delta<\lambda$, where δ is the modulus of uniform convergence arbitrarily specified in Lemma 4. Solving for ζ_0 in the equation of (8) gives $\zeta_0=\pi/(U(0)+1)$. Then, using the definition of λ above, $\zeta_0=k\pi/(1+k-\lambda)$. Since $\delta>0$ is arbitrary and λ may be chosen as small as we please such that $\delta<\lambda$ and $\zeta_2-\zeta_1$ may be taken as small as we please, then $\theta_0\approx\pi k/(k+1)$ with any degree of accuracy desired.

Hence the angular measure of the arc Γ is $2\pi k/(k+1)$, since by letting $e^{i\gamma(\theta(\varepsilon))}$ be the image of $w=(1+\varepsilon)e^{i\theta(\varepsilon)}$ under the transformation $\varphi(w, X^*(\varepsilon))$, then the fact that $\lim_{\varepsilon\to 0}\varphi(w, X^*(\varepsilon))=w$ uniformly on $|w|\leqq R_1$ implies that $\lim_{\varepsilon\to 0}\gamma(\theta(\varepsilon))=\theta_0 \pmod{2\pi}$. This completes the proof of Theorem 1.

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