PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

MIXTURES OF NONATOMIC MEASURES

K. P. S. BHASKARA RAO AND M. BHASKARA RAO

ABSTRACT. In this paper, we answer the following question: Is a mixture of nonatomic measures nonatomic? The answer turns out to be no, in general. Four sufficient conditions are given under which mixtures become nonatomic.

1. **Introduction and definitions.** Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two Borel structures. Let P be a transition probability on $X \times \mathscr{B}$, i.e., P is a function defined on $X \times \mathscr{B}$ taking values in $[0, 1]$ such that $P(x, \cdot)$ is a probability measure on $\mathscr{B}$ for every x in X and $P(\cdot, B)$ is an $\mathscr{A}$-measurable function for every B in $\mathscr{B}$. Let γ be any probability measure on $\mathscr{A}$ and define $\mu(B)=\int P(x, B)\, \gamma(dx)$ for B in $\mathscr{B}$. μ is a probability measure on $\mathscr{B}$, and is called the mixture of P with respect to γ. P is said to be nonatomic if $P(x, \cdot)$ is nonatomic for every x in X.

§2 gives an example of a nonatomic P and a measure γ such that the mixture of P with respect to γ is not nonatomic. §3 gives four sufficient conditions under which all mixtures become nonatomic.

All measures considered in this paper are finite. A measure γ on $\mathscr{A}$ is said to be two-valued if there is a real number a, positive, such that $\gamma(A)=0$ or a for every A in $\mathscr{A}$ and $\gamma(X)=a$. The terminology used here follows closely that of Neveu [**2**].

2. **Example.** Let X be any uncountable set and $\mathscr{A}$, the countable cocountable σ-algebra on X. For each x in X, let $(Y_x, \mathscr{B}_x, \mu_x)$ be a nonatomic probability space. Let $Y=\prod_{x\in X} Y_x$, the product space and $\mathscr{B}=\prod_{x\in X} \mathscr{B}_x$, the product σ-algebra. Fix f_0 in Y. Let $f_0^x=f_0/X-\{x\}$, the restriction of f_0 to $X-\{x\}$. For every B in $\mathscr{B}$, define $B_x=f_0^x$-th section of $B=\{g(x)\in Y_x : g\in B$ and $g=f_0$ on $X-\{x\}\}$. It is easy to verify that $\{B_x : B\in\mathscr{B}\}=\mathscr{B}_x$. Now, $P: X\times\mathscr{B}$ into $[0, 1]$ is defined as follows. $P(x,B)=\mu_x(B_x)$. For every x in X, $P(x, \cdot)$ is a nonatomic probability measure on $\mathscr{B}$. Since every B in $\mathscr{B}$ is a countable-dimensional cylinder, it follows that $B_x=\varnothing$ for all but a countable number of x's or $=Y_x$ for all but a countable number of x's. Consequently, $P(x, B)=0$ for all but a

Received by the editors April 22, 1971 and, in revised form, June 23, 1971.

AMS 1969 *subject classifications*. Primary 2813, 2815; Secondary 6060.

Key words and phrases. Atom of a σ-algebra, atom of a measure, nonatomic measure, transition probability, mixture, separable and Lindelöf topological spaces.

countable number of x's or $=1$ for all but a countable number of x's. Hence, $P(\cdot, B)$ is $\mathscr{A}$-measurable for every B in $\mathscr{B}$. Let γ be the 0-1 valued measure on $\mathscr{A}$ defined by $\gamma(A)=0$ or 1 according as A is countable or cocountable. The mixture μ of P with respect to γ is 0-1 valued and hence cannot be nonatomic. In fact, μ is a degenerate measure (at f_0).

3. THEOREM 1. *If $\mathscr{B}$ is separable, i.e., has a countable generator, then the mixture μ of any nonatomic transition probability P with respect to any probability measure γ is nonatomic.*

PROOF. Since $\mathscr{B}$ is separable, any measure on $\mathscr{B}$ is nonatomic if and only if it is continuous. (A measure on $\mathscr{B}$ is said to be continuous if the measure of any atom of $\mathscr{B}$ is zero.) This can be proved by noting that there are no two-valued continuous measures on any separable σ-algebra. Since a mixture of continuous measures is continuous, our theorem follows.

THEOREM 2. *If $\{P(x, \cdot): x \text{ in } X\}$ is a dominated family of nonatomic measures, i.e., there exists a σ-finite measure φ on $\mathscr{B}$ such that $P(x, \cdot) \ll \varphi$ for every x in X, then any mixture μ of P is nonatomic.*

PROOF. By a theorem of Halmos and Savage (see, for example, [**2**, p. 112]), the family $\{P(x, \cdot): x \text{ in } X\}$ is equivalent to a countable subfamily $\{P(x_n, \cdot): n \geqq 1\}$, i.e., $P(x, B)=0$ for every x in X if and only if $P(x_n, B)=0$ for every $n \geqq 1$. If $\tau(\cdot)=\sum_{n \geqq 1} (1/2^n) P(x_n, \cdot)$, then the family $\{P(x, \cdot): x \text{ in } X\}$ is equivalent to τ. By direct argument, it follows that τ is nonatomic. Further, the mixture μ is dominated by τ. Hence, μ is nonatomic. (See, for example, [**1**, Theorem 2.4, p. 653].)

THEOREM 3. *Let X be a topological space having a countable dense set and $\mathscr{A}$ be the σ-algebra generated by open subsets of X. Further, assume that $P(\cdot, B)$ is a continuous function on X for every B in $\mathscr{B}$. If P is nonatomic, then any mixture of P is nonatomic.*

In order to prove this theorem, we need the following lemmas.

LEMMA 1. *Let φ be a nonatomic measure on $\mathscr{B}$ and τ be a two-valued measure on $\mathscr{B}$. Then φ and τ are mutually singular.*

PROOF. By the Lebesgue decomposition theorem, we can write $\varphi=\varphi_1+\varphi_2$ where $\varphi_1 \ll \tau$, and φ_2 and τ are mutually singular. Since τ is two-valued, φ_1 is also two-valued. As $\varphi_1 \leqq \varphi$, φ_1 is also nonatomic. (See, for example, [**1**, p. 653].) Hence, $\varphi_1=0$, $\varphi_2=\varphi$ and so φ and τ are mutually singular.

LEMMA 2. *If $\varphi_1, \varphi_2, \cdots$ is a sequence of nonatomic measures and τ is a two-valued measure on $\mathscr{B}$, then there exists B in $\mathscr{B}$ such that $\tau(B)=\tau(Y)$ and $\varphi_i(B)=0$ for every $i \geqq 1$.*

PROOF. Let $\varphi(\cdot)=\sum_{n\geqq 1}(1/2^n)(1/\varphi_n(Y))\varphi_n(\cdot)$. Then, φ is a nonatomic measure and hence by Lemma 1, φ and τ are mutually singular. So, there exists a B in $\mathscr{B}$ such that $\tau(B)=\tau(Y)$ and $\varphi(B)=0$. This implies $\varphi_i(B)=0$ for every $i\geqq 1$.

PROOF OF THEOREM 3. Let μ be a mixture of P with respect to γ. Suppose μ is not nonatomic. Let B_0 be an atom of μ. Since $\mu(B_0)>0$, the open set $U=\{x \text{ in } X: P(x, B_0)>0\}$ has positive γ-measure. As X contains a dense denumerable set, U contains a dense denumerable set, say, x_1, $x_2, \cdots$. The measure μ on $B_0\cap\mathscr{B}$ is two-valued. $P_1(x_n, \cdot)=P(x_n, \cdot)/B_0$; the restriction of $P(x_n, \cdot)$ to $B_0\cap\mathscr{B}$ is a sequence of nonatomic measures on $B_0\cap\mathscr{B}$. By Lemma 2, there exists a B in $\mathscr{B}$, $B\subset B_0$ such that $\mu(B)=\mu(B_0)$ and $P(x_n, B)=0$ for every $n\geqq 1$. Let $C=\{x \text{ in } X: P(x, B)=0\}$. C is a closed set and each x_n is in C. So, $U\subset$ Closure of $\{x_n: n\geqq 1\}\subset C$. Now,

$$0 \neq \mu(B_0) = \mu(B) = \int P(x, B)\,\gamma(dx)$$
$$= \int_U P(x, B)\,\gamma(dx) + \int_{U^c} P(x, B)\,\gamma(dx) = 0 + 0 = 0, \quad \text{a contradiction.}$$

That the second term is zero follows from the fact

$$P(x, B) \leqq P(x, B_0) = 0 \quad \text{if } x \text{ is in } U^c.$$

THEOREM 4. *Let X be a Lindelöf topological space and $\mathscr{A}$ be the σ-algebra on X generated by open subsets of X. Further, assume that $P(\cdot, B)$ is a continuous function on X for every B in $\mathscr{B}$. If P is nonatomic, then any mixture of P is nonatomic.*

PROOF. Let μ be a mixture of P with respect to γ. Let $B_0\in\mathscr{B}$ be such that $\mu(B_0)$ is positive. Then $U=\{x \text{ in } X: P(x, B_0)>0\}$ is an open set of positive γ-measure. Write $U=\bigcup_{n\geqq 1} U_n$, where

$$U_n = \{x \text{ in } X: P(x, B_0) \geqq 1/n\}.$$

Since $\gamma(U)$ is positive, there exists N such that $\gamma(U_N)$ is positive. Since U_N is closed, it is Lindelöf. For every $B\subset B_0$, define

$$U_B = \{x \text{ in } X: P(x, B_0) > P(x, B) > 1/N + 1\}.$$

U_B is an open set for every $B\subset B_0$, B in $\mathscr{B}$. Now, $\{U_B: B\subset B_0 \text{ and } B \text{ in } \mathscr{B}\}$ is an open cover of U_N. For, let x be in U_N. Then $P(x, B_0)\geqq 1/N> 1/N+1$. By nonatomicity, there exists a C in $\mathscr{B}$, $C\subset B_0$ such that $P(x, B_0)> P(x, C)>1/N+1$. Consequently, x is in U_C. By the Lindelöf property of U_N, we can find a countable subcover for U_N from $\{U_B: B\subset B_0 \text{ and } B \text{ in } \mathscr{B}\}$.

Hence, there exists $B \subset B_0$, B in $\mathscr{B}$ such that $\gamma(U_B)$ is positive. On U_B, $1/N+1 < P(x, B) < P(x, B_0)$. So, $0 < \mu(B) < \mu(B_0)$. This proves the nonatomicity of μ.

REMARKS. (1) For the nonatomicity of μ, the first theorem lays conditions on the Borel structure $(Y, \mathscr{B})$, the second on the family $\{P(x, \cdot): x \text{ in } X\}$ and the third and fourth on $(X, \mathscr{A})$ and on the family $\{P(\cdot, B): B \text{ in } \mathscr{B}\}$ of functions.

(2) For a related result see Theorem 1.2 of [**1**, p. 651].

(3) If one wishes to have a counterexample in the setup of a single Borel structure, one can construct one using the one given in §2.

The authors are thankful to Mr. D. Ramachandran for useful discussions regarding §3.

REFERENCES

1. R. A. Johnson, *Atomic and nonatomic measures*, Proc. Amer. Math. Soc. **25** (1970), 650–655.

2. J. Neveu, *Bases mathématiques du calcul des probabilités*, Masson, Paris, 1964; English transl., Holden-Day, San Francisco, Calif., 1965. MR **33** #6659; MR **33** #6660.

INDIAN STATISTICAL INSTITUTE, CALCUTTA 35, INDIA