ON COMPACTNESS IN FUNCTION SPACES

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ABSTRACT. There are shown some implications from pseudo-compactness to compactness or sequential compactness. The latter, sequential compactness, is obtained via metrization.

In [4] there are a number of criteria for function spaces and topological vector spaces showing that countable compactness implies compactness or sequential compactness. Since pseudocompactness weakens countable compactness, it seems natural to ask whether pseudocompactness leads to the same implications. For example, in [8] it is proved that in a complete locally convex space a weakly pseudocompact subset is relatively weakly compact. The aim of this note is to give some positive answer to the above question. We shall only deal with the topology of point-wise convergence, since one can easily deduce from our results similar ones for the topology of uniform convergence on compacta or more general subsets of the domain.

NOTATIONS. We shall denote by

$C_p(X, Y)$ the topological space of all continuous maps $X \to Y$ with the topology of pointwise convergence,

$C_p(X)$ the space $C_p(X, R)$.

1. Compactness.

1.1 THEOREM. *Let X be a topological space with the following property*:

(∗) *$f: X \to R$ is continuous iff, for each countable subset C of X, $f|_C$ is continuous.*

Let Y be a uniformizable space and $H \subseteq C(X, Y)$. Then H is relatively compact in $C_p(X, Y)$ iff H is relatively pseudocompact and, for every $x \in X$, $\mathrm{cl}(H(x))$ is complete for the fine uniformity of Y.

Recall that a topological space X is said to be *pseudocompact* iff every continuous map $X \to R$ is bounded, while $A \subseteq X$ is said to be *relatively pseudocompact* iff $\bar{A}$ is pseudocompact. The *fine uniformity* is the finest compatible uniformity.

Examples of spaces X with property (∗) are: first countable spaces and (by [7] and a remark at [3, pp. 789–790]) products of separable metric spaces. Moreover, any separable (=there is a countable dense subset) space, hence any sum of separable spaces, has property (∗), as one can easily see by a well-known theorem on extension from dense subsets of continuous mappings into regular spaces. Therefore we have the following extension of [7, II] to noncomplete spaces X's:

1.2 COROLLARY. *Let X be any locally convex space. If X' is weakly∗ separable, then for the weak topology of X relative compactness and relative pseudocompactness are the same.*

Finally, the above theorem generalizes [4, Theorem 2] to noncompact domains, since the closure of a relatively countably compact subset is pseudocompact. Note that our proof is somewhat simpler than the proof of [4, Theorem 2].

PROOF OF 1.1. The necessity being clear, we prove only the sufficiency.

Case 1. Y is Hausdorff (hence completely regular). Let $\bar{H}$ be the closure of H in $C_p(X, Y)$. Since $\bar{H}(x) = \{f(x) \mid f \in \bar{H}\}$ is a continuous image of $\bar{H}$, $\bar{H}(x)$ is pseudocompact. Since $\bar{H}(x)$ is dense in $\mathrm{cl}(H(x))$, $\mathrm{cl}(H(x))$ is pseudocompact. Consequently $\mathrm{cl}(H(x))$ is compact by the precompactness of compatible uniformities on a pseudocompact space. Therefore $\prod_{x \in X} \mathrm{cl}(H(x))$ is compact by Tychonoff theorem. Then we have only to show that each f in the closure H^* of H in $\prod_{x \in X} \mathrm{cl}(H(x))$ is a continuous map $X \to Y$. Since Y is completely regular, Y is embeddable into a product $\prod_{\alpha \in A} R$ of copies of the reals. Then: $f: X \to Y$ is continuous iff $p_\alpha \circ f$, p_α being the canonical projection $\prod_\alpha R \to R$, is continuous for every $\alpha \in A$. Choose $f_0 \in H^*$, and $\alpha \in A$. The subspace $H_\alpha = \{p_\alpha \circ f \mid f \in \bar{H}\}$ of $C_p(X)$ is pseudocompact since it is a continuous image of $\bar{H}$. By (∗), $p_\alpha \circ f_0$ is continuous iff, for each countable $C \subseteq X$, $p_\alpha \circ f \mid_C$ is continuous. Choose then a countable $C \subseteq X$. Let

$$r: \prod_{x \in X} R \to \prod_{x \in C} R$$

be the projection $f \rightsquigarrow f \mid_C$. Since r is continuous, $r(H_\alpha)$ is a pseudocompact subspace of $C_p(C)$. Since C is countable, $C_p(C)$ is metrizable. Since a metrizable pseudocompact space is compact, $r(H_\alpha)$ is compact. Then $r(H_\alpha)$ is a closed subset of $\prod_{x \in C} R$. Therefore $r(p_\alpha \circ f_0) \in r(H_\alpha)$, so that there is $g \in H_\alpha$ such that $g(x) = p_\alpha(f_0(x))$ for all $x \in C$. Thus $p_\alpha \circ f_0 \mid_C$ is continuous.

Case 2. Y is not Hausdorff. Let $\sim$ be the equivalence relation

$$y' \sim y'' \Leftrightarrow \overline{\{y'\}} = \overline{\{y''\}}$$

on Y. Let $Y/\!\!\sim$ be the quotient space and $q: Y \to Y/\!\!\sim$ the quotient map.

Since $H_0 = \{q \circ f \mid f \in H\}$ is a continuous image of H, H_0 is a relatively pseudo-compact subspace of $C_p(X, Y/\sim)$. Since $Y/\sim$ is completely regular, H_0 is relatively compact by Case 1. This implies the conclusion by the openness of the map q. Q.E.D.

2. **Sequential compactness.** We give now some results on the equivalence between pseudocompactness and sequential compactness. Since the closure of a relatively countably compact subset is pseudocompact, the following theorem generalizes [4, Theorem 4] and [6, §18, Problem L]. Note that [4, Theorem 4] fails without some additional hypothesis on the range: a counterexample is given by taking constant maps into a countably compact space which is not sequentially compact. The assumptions on domains of the following results cannot longer be weakened since there are completely regular spaces which are countably compact but not sequentially compact, and every completely regular space is embeddable into a space $C_p(C_p(X))$.

According to [2, §2, Exercise 23], a topological space is called *submetrizable* iff its topology is finer than a metrizable topology. We shall say that a topological space is σ-pseudocompact iff it is the union of a sequence of pseudocompact subsets. Examples of submetrizable completely regular nonmetrizable spaces are: spaces of type $C_p(X, Y)$ with X separable, *LF*-spaces.

2.1 THEOREM. *If X is a topological space with a dense σ-pseudocompact subset and Y a submetrizable completely regular space, then every separable pseudocompact subset H of $C_p(X, Y)$ is metrizable.*

PROOF. Let $(E_n)_{n=1}^{\infty}$ be a sequence of pseudocompact subsets of X whose union is dense in X, and let d be a metric whose topology is less fine than the topology of Y. By a well-known theorem of Kuratowski (Y, d) is embeddable into a Banach space B, and we identify the maps $X \to Y$ with their natural corresponding maps $X \to B$. Let Z be the topological vector subspace of $C_p(X, B)$ spanned by H. Since H is a separable subspace of $C_p(X, Y)$, Z is separable: let $\{f_n \mid n \in Z^+\}$ be dense in Z. The weak topology τ_n induced by $\{f_i|_{E_n} \mid n \in Z^+\}$ is pseudometrizable and less fine than the topology of E_n, hence bicompact by a well-known property of pseudocompactness. Then by [4, p. 175, footnote], $f|_{E_n} : (E_n, \tau_n) \to B$ is continuous for all $f \in Z$. Since every (E_n, τ_n) is separable and $\bigcup_{n=1}^{\infty} E_n$ dense in X, we conclude that there is a countable subset E of X such that

$$f|_E = g|_E \Rightarrow f = g \qquad (f, g \in Z).$$

Consequently the origin of Z is a G_δ, being the intersection of its neighborhoods

$$\{f \in Z \mid |f(x)| < 1/n\} \qquad (x \in E; n \in Z^+).$$

Then by a well-known property of neighborhoods based of vector topologies, Z is submetrizable, so that the subspace H of Z is metrizable by [2, §2, Exercise 23-b)]. Again by [2, §2, Exercise 23-b)], B induces on $H(x)$, $x \in X$, the same topology as Y, so that H receives the same topologies from $C_p(X, B)$ and $C_p(X, Y)$.   Q.E.D.

The following result may be added to a long list (see [5, Chapter vii]) of results which lead to conclude that products of separable spaces are in some way "almost separable".

2.2 COROLLARY.   *If X is an open subset of an arbitrary product $\prod_{\alpha \in A} X_\alpha$ of separable spaces and Y a submetrizable completely regular space, then any separable pseudocompact subset of $C_p(X, Y)$ is metrizable.*

PROOF.   Let H be a countable, relatively pseudocompact subset of $C_p(X, Y)$, and let $\bar{H}$ be its closure in $C_p(X, Y)$. Let us denote, for $B \subseteq A$, by π_B the projection $(x_\alpha)_{\alpha \in A} \leadsto (x_\alpha)_{\alpha \in B}$ restricted to X. By Gleason's theorem [5, vii. 23], to each $f \in H$ there corresponds a countable subset $C_f \subseteq A$ such that

$$f = g_f \circ \pi_{C_f},$$

where $g_f : \pi_{C_f}(X) \to Y$ is continuous. Define $C = \bigcup_{f \in H} C_f$. Since H is countable, also C is. The equality $f = g_f \circ \pi_{C_f}$ means that if two points $x, y \in X$ have the same coordinates for indices in C_f, then $f(x) = f(y)$. Consequently, if $x, y \in X$ have the same coordinates for indices in C, then $f(x) = f(y)$ for all $f \in H$, i.e.

$$f = g_f \circ \pi_C \qquad (f \in H),$$

where $g_f : \pi_C(X) \to Y$ is continuous because π_C is an open map (being the restriction on an open set of an open map). Choose $h \in \bar{H}$. There is a net $(f_\beta)_\beta$ in H converging pointwise to h. Consider any $x, y \in X$ having the same coordinates in C. Then $f_\beta(x) = f_\beta(y)$ for all β. Hence

$$h(x) = \lim_\beta f_\beta(x) = \lim_\beta f_\beta(y) = h(y)$$

since limits are unique in a Hausdorff space. Therefore we may repeat a reasoning made above to obtain

$$h = g_h \circ \pi_C$$

with g_h continuous. Then $h \leadsto g_h$ is a map $\varphi : \bar{H} \to C_p(\pi_C(X), Y)$. Obviously φ is a homeomorphism into. By 2.1, $\varphi(\bar{H})$ is metrizable since $\pi_C(X)$ is separable. Consequently $\bar{H}$ is metrizable.   Q.E.D.

2.3 REMARKS.   (i) Concerning the sole equivalence between countable compactness and sequential compactness, 2.1 and 2.2 may be proved under the hypothesis "Each point of Y is a G_δ". This needs a result well known for compact spaces: *A pseudocompact uniformizable space whose*

points are G_δ-sets is 2nd countable. (Its proof is somewhat more difficult than in case of compact spaces.)

(ii) The argument in the proof of 2.2 suggests a proof of the following result:

If X is a paralindelöfian space and Y a metric space, then every pseudocompact equicontinuous subset H of $C_p(X, Y)$ is metrizable,

(the paralindelöfian spaces being defined in [1] and characterized topologically in [9]). We only briefly sketch the proof since this result is not entirely satisfactory by the assumption of equicontinuity of H. By [5, iii. 38], H is equi-uniformly continuous with respect to the fine uniformity μ of X and the metric of Y. Since μ has a base made of countable uniform covers, μX is embeddable into a product of separable metric spaces. Now we repeat the argument in the proof of [10, Theorem] to factor each $f \in H$ into $f = g_f \circ \pi_C$ with π_C a fixed countable projection and $\{g_f | f \in H\}$ equi-uniformly continuous. Then we conclude as in the case of 2.2.

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