A LOWER BOUND FOR THE PERMANENT OF A (0, 1)-MATRIX

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Abstract. Let \( A = (a_{ij}) \) be an \( n \times n \) fully indecomposable (0, 1)-matrix. It is shown that if each row sum of \( A \) is at least \( k \) then

\[
\text{per } A \geq \sum_{i=1}^{n} a_{ii} - 2n + 2 + \sum_{m=1}^{k-1} (m! - 1)
\]

This improves an inequality obtained by H. Minc.

Lower bounds for the permanent of a (0, 1)-matrix are of considerable combinatorial interest. A well-known theorem of M. Hall [1] states that if \( A \) is an \( n \times n \) (0, 1)-matrix with positive permanent and at least \( k \) positive entries in each row, then

\[
\text{per } A \geq \sigma(A) - 2n + 2,
\]

where \( \sigma(A) \) is the sum of all entries of \( A \). Recently H. Minc [3] proved that if \( A \) is an \( n \times n \) fully indecomposable (0, 1)-matrix then

\[
\text{per } A \geq \sigma(A) - 2n + 2,
\]

where \( \sigma(A) \) is the sum of all entries of \( A \). In this note we show that Hall’s inequality can be used to improve Minc’s inequality.

Let \( A = (a_{ij}) \) be an \( n \times n \) matrix. Let \( A^{km} \) be the \( (n-1) \times (n-1) \) submatrix of \( A \) that remains after row \( k \) and column \( m \) are removed. If \( A \) contains an \( s \times (n-s) \) zero submatrix, for some \( 1 \leq s \leq n-1 \), then \( A \) is partly decomposable; otherwise, \( A \) is fully indecomposable. If \( A \) is fully indecomposable, then \( A^{km} \) is partly decomposable whenever \( a_{km} \neq 0 \), then \( A \) is nearly decomposable.

Theorem. If \( A \) is an \( n \times n \) fully indecomposable (0, 1)-matrix with \( r(A) \geq k \), then

\[
\text{per } A \geq \sigma(A) - 2n + 2 + \sum_{m=1}^{k-1} (m! - 1).
\]

Proof. The proof is by induction on \( k \). If \( k = 1 \) or 2, then this statement follows from Minc’s inequality. Suppose that it holds for all \( l < k \), where \( k \geq 3 \), and let \( A = (a_{ij}) \) be an \( n \times n \) fully indecomposable (0, 1)-matrix with \( r(A) \geq k \). Since \( r(A) \geq 3 \), it follows from Hartfiel’s form for
nearly decomposable matrices [2] that \( A \) is not nearly decomposable. Hence, there exist \( i, j \in \{1, \cdots, n\} \) such that \( a_{ij} = 1 \) and \( A^{ij} \) is fully indecomposable. Since \( r(A^{ij}) \geq k-1 \) and \( \sigma(A^{ij}) = \sigma(A) - 1 \), the inductive assumption implies that

\[
\text{per } A^{ij} \geq \sigma(A) - 1 - 2n + 2 + \sum_{m=1}^{k-2} (m! - 1).
\]

Since \( A \) is fully indecomposable, \( \text{per } A > 0 \). Hence, since \( r(A_{ij}) \geq k-1 \), Hall's inequality implies that

\[
\text{per } A_{ij} \geq (k - 1)!. \tag{3}
\]

Since \( a_{ij} = 1 \), \( \text{per } A = \text{per } A^{ij} + \text{per } A_{ij} \). Combining this with (2) and (3), we have (1).

Let \( \Lambda_n^k \) be the set of all \( n \)-square \((0, 1)\)-matrices with precisely \( k \) positive entries in each row and each column. Minc [3] showed that if \( A \in \Lambda_n^k \) then \( \text{per } A \geq n(k - 2) + 2 \). Hartfiel [2] discovered that if \( A \in \Lambda_n^4 \) then \( \text{per } A \geq n + 3 \). Using our theorem it is easy to prove the following.

**Corollary.** If \( A \in \Lambda_n^k \) then

\[
\text{per } A \geq n(k - 2) + 2 + \sum_{m=1}^{k-2} (m! - 1).
\]

**References**


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