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REGULAR RINGS AND INTEGRAL EXTENSION OF A REGULAR RING

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ABSTRACT. In this paper we show that a ring (not necessarily commutative) with identity element and without nonzero nilpotent elements is a von Neumann regular ring if every completely prime ideal is a maximal right ideal. Using this result, we show an integral extension (not necessarily commutative) without nonzero nilpotent elements of a regular ring is itself a regular ring.

Let R be a ring with unit 1. A proper ideal P in R is called a completely prime ideal if $ab \in P$ implies either $a \in P$ or $b \in P$. A proper ideal Q in R is called a semicompletely prime ideal if $a^n \in Q$, for some integer n, then $a \in Q$. Andrunakievič and Rjabuhin [1] proved that if Q is a semicompletely prime ideal and M^* is a maximal m-system where $M^* \cap Q = \varnothing$, then the complement of M^* in R is a completely prime ideal.

For any $x \neq 0$ in R, let $J(x)$ be the right annihilator ideal of x in R and $A(xR) = \{y \in R \mid \text{there exists } r \in R,\ xr \neq 0 \text{ and } (xr)y = 0\}$. $A(xR)$ is the set of zero-divisors on the right R-module xR. $J(x) \subset A(xR)$ and the complement of $A(xR)$ in R is a multiplicatively closed system.

LEMMA 1. *If R has no nonzero nilpotent elements then $J(x)$ is a semicompletely prime ideal and $ab \in J(x)$ implies $ba \in J(x)$.*

PROOF. In a ring without nonzero nilpotent elements, $yz=0$ implies $zy=0$. Since $(zy)^2 = z(yz)y$. Therefore $J(x)$ is also the left annihilator ideal of x and hence an ideal in R. If $b^n \in J(x)$, for some integer $n > 1$, then $0 = xb^n = xb^{n-1}b = bxb^{n-1}$. If $n=2$ then $bxbx=0$. This implies $bx=0$ and $b \in J(x)$. If $n>2$ then $0 = bxb^{n-2}b$ and hence $(bxb^{n-2})^2 = 0$. Thus $bxb^{n-2} = xb^{n-1} = 0$. Continuing this process, after a finite number of steps, we obtain $xb=0$. $J(x)$ is a semicompletely prime ideal. If $ab \in J(x)$ then $(ba)^2 = b(ab)a \in J(x)$ and hence $ba \in J(x)$.

THEOREM 1. *Let R be a ring with no nonzero nilpotent elements. For any $x \neq 0$ in R, if P is a minimal completely prime ideal containing $J(x)$ then $P \subset A(xR)$.*

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PROOF. Let $M=\{(a_1m_1)\cdots(a_nm_n)|n$ a natural number, $a_i\notin P$, $m_i\notin A(xR)\}$. M is a multiplicatively closed system and contains the complement of P and the complement of $A(xR)$. $M\cap J(x)=\varnothing$. Since if $(a_1m_1)\cdots(a_{n-1}m_{n-1})(a_nm_n)\in(M\cap J(x))$ and $n>1$ then

$$(a_1m_1)\cdots(a_{n-1}m_{n-1})a_n\in J(x).$$

By Lemma 1, $((a_na_1)m_1)\cdots(a_{n-1}m_{n-1})\in J(x)$. $(a_na_1)\notin P$ because P is completely prime. Continuing this process, eventually we will obtain an $am_1\in(J(x)\cap M)$ where $a\notin P$. But this is impossible, since $(xa)m_1=0$ implies $xa=0$. $a\in J(x)\subset P$. Let M^* be a maximal m-system where $M\subset M^*$ and $M^*\cap J(x)=\varnothing$. By [**1**], $C(M^*)=P^*$, the complement of M^*, is a completely prime ideal. $J(x)\subset P^*\subset P$ and $P^*\subset A(xR)$. By the minimality of P, $P=P^*\subset A(xR)$.

LEMMA 2. *Let R be a ring with unit* 1 *and no nonzero nilpotent elements. If every completely prime ideal of R is a maximal right ideal then every nonunit element of R is contained in a completely prime ideal.*

PROOF. If x is not a unit in R then $xR\neq R$, because a right inverse of an element is the inverse of the element in a ring with no nonzero nilpotent elements. Let $M=\{xr+1|r\in R\}$. M is a multiplicatively closed system and $0\notin M$. Since the zero ideal is semicompletely prime, there exists a completely prime ideal P where $P\cap M=\varnothing$. If $xR\not\subset P$, then $xR+P=R$. $xr+p=1$ for some $r\in R$ and some $p\in P$. $p=x(-r)+1$ is in M, a contradiction.

THEOREM 2. *If R has no nonzero nilpotent elements and every completely prime ideal is a maximal right ideal, then any nonzero-divisor is a unit.*

PROOF. If $x\in R$ with $J(x)=0$, then by Theorem 1, any completely prime ideal consists of zero-divisors. By the above lemma, x must be a unit.

A ring is called a regular ring (von Neumann), if for each x there exists y such that $xyx=x$. In a regular ring, it is easy to show every completely prime ideal is a maximal right ideal. A commutative ring is a regular ring if and only if it has no nonzero nilpotent elements and every prime ideal is a maximal ideal [**2**, p. 63].

THEOREM 3. *Let R be a ring with unit* 1 *and no nonzero nilpotent elements. If every completely prime ideal of R is a maximal right ideal then R is a regular ring. Furthermore, for each* $x\in R$ *there exists a unit u such that* $x^2u=x$.

PROOF. $x\neq 0$ in R, $R/J(x)$ is again a ring without nonzero nilpotent elements and every completely prime ideal is a maximal right ideal.

$\bar{x}=x+J(x)$ is not a zero-divisor in $R/J(x)$. Since $xb \in J(x)$ implies $bx \in J(x)$, $(xb)^2=xbxb=0$ implies $b \in J(x)$. By Theorem 2, there exists $r \in R$ and $p \in J(x)$, $xr+p=1$. $xrx=x^2r=x$. R is a regular ring.

In a ring with no nonzero nilpotent elements, every idempotent is central. Let $e=xr$ where $x^2r=x$ and $u=(1-e)+er$. It is easy to verify $J(u)=0$ and $x^2u=x$. By Theorem 2, u is a unit.

COROLLARY 1. *Let R be a ring with unit* 1 *and no nonzero nilpotent elements. R is a regular ring if R/P is regular for every prime ideal P in R.*[1]

PROOF. Let Q be a completely prime ideal in R. Q is prime. R/Q is a division ring, because R/Q is von Neumann regular and has no nonzero-divisor. Therefore Q is a maximal right (and left) ideal in R and R is a regular ring.

COROLLARY 2. *A commutative ring with a unit is a regular ring if and only if it has no nonzero nilpotent elements and every prime ideal is a maximal ideal.*

Integral extension. S is a ring with identity element and R is a subring of S containing the identity element of S. We call S an integral extension of R, or S is integral over R, if for each $s \in S$, there exist a positive integer n and elements $a_{n-1}, \cdots, a_0$ in R such that $s^n+a_{n-1}s^{n-1}+\cdots+a_0=0$.

THEOREM 4. *Let S be an integral extension of R. If R is a regular ring then any completely prime ideal Q in S is a maximal right ideal in S.*

PROOF. Let $P=Q\cap R$. P is a completely prime ideal in R. Since R is a regular ring, R/P is a division ring. S/Q is an integral domain (not necessarily commutative) and is an integral extension of R/P. S/Q must be a division ring and hence Q is a maximal right ideal in S.

Together with Theorem 3, we have:

THEOREM 5. *An integral extension with no nonzero nilpotent elements of a regular ring is itself a regular ring.*

REFERENCES

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2. I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, Mass., 1970. MR **40** #7234.

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[1] This result was discovered by I. N. Herstein. Herstein proves it from a different approach and without assuming the existence of an identity element.