HEREDITARY RADICALS IN JORDAN RINGS

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Abstract. The object of this paper is to examine some radical properties of quadratic Jordan algebras and to show that under certain conditions, $R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J})$ where $\mathfrak{B}$ is an ideal of a quadratic Jordan algebra $\mathfrak{J}$, $R(\mathfrak{B})$ is the radical of $\mathfrak{B}$, and $R(\mathfrak{J})$ is the radical of $\mathfrak{J}$.

1. Preliminaries. We adopt the notation and terminology of an earlier paper [2] concerning quadratic Jordan algebras (defined by the quadratic operators $U_x$) as opposed to linear Jordan algebras (defined by the linear operators $L_x$). Thus we have a product $U_{xy}$ linear in $y$ and quadratic in $x$ satisfying the following axioms as well as their linearizations:

(UQJ I) $U_1 = I$ (1 the unit);
(UQJ II) $U_{U(x,y)} = U_x U_y U_{xy}$;
(UQJ III) $U_x V_{y,z} = V_{y,z} U_x (V_{y,z} = (xyz) = U_x y)$.

Throughout this paper $\mathfrak{J}$ will denote a quadratic Jordan algebra over an arbitrary ring of scalars $\Phi$.

Define a property $R$ of a class of rings (e.g. associative rings or Jordan rings) to be a radical property if it satisfies the following three conditions [1]:

(a) Every homomorphic image of an $R$ ring is again an $R$ ring.
(b) Every ring $\mathfrak{J}$ contains an $R$ ideal $R(\mathfrak{J})$ which contains every other $R$ ideal of $\mathfrak{J}$. The maximal $R$ ideal $R(\mathfrak{J})$ is called the $R$ radical of $\mathfrak{J}$.
(c) For $\mathfrak{B}$ an ideal of $\mathfrak{J}$, if $\mathfrak{B}$ and $\mathfrak{J} \mathfrak{B}$ are $R$ rings, then so is $\mathfrak{J}$.

An immediate consequence of this definition is $R(\mathfrak{B}/R(\mathfrak{B})) = 0$. If $R(\mathfrak{B}) = 0$, $\mathfrak{B}$ is said to be $R$ semisimple.

Many well-known radical properties, but not all, satisfy a further condition:

(d) Every ideal of an $R$ ring is again an $R$ ring (i.e. property $R$ is inherited by ideals of an $R$ ring).

If a radical property satisfies condition (d) that property is called hereditary.

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1 The results of this paper are included in the author's doctoral dissertation written at the University of Virginia.
In this paper we shall consider all radical properties R of \( \mathcal{J} \) such that if \( \mathcal{J} \) contains no R ideals then \( \mathcal{J} \) contains no absolute zero divisors (where \( z \in \mathcal{J} \) is an absolute zero divisor if \( U_z = 0 \)). Henceforth such radical properties will be called radical properties of type A.

Two of the more prominent radical properties are quasi-invertibility and nil: An element \( z \) belonging to a Jordan algebra \( \mathcal{J} \) is quasi-invertible (q.i.) with quasi-inverse \( w \in \mathcal{J} \) if \( 1 - z \) is invertible with inverse \( 1 - w \) in \( \Phi 1 \otimes \mathcal{J} \). A subset is called quasi-invertible if all of its elements are quasi-invertible in \( \mathcal{J} \). The maximal quasi-invertible ideal of an algebra is usually called the Jacobson radical and plays an important role in the structure theory. If we define an element \( z \in \mathcal{J} \) to be nilpotent if \( z^n = 0 \) for some \( n \) (here powers of an element are defined recursively by \( z^0 = 1, z^1 = z, z^{n+2} = U_z z^n \)), then a nil ideal is one all of whose elements are nilpotent. There is a unique maximal nil ideal containing all other nil ideals and this is called the nil radical. In [3] McCrimmon shows that nil and quasi-invertibility are radical properties of type A.

A lesser known radical property is that of antiprime: An algebra \( \mathcal{J} \) is strongly semiprime if it contains no absolute zero divisors. An algebra \( \mathcal{J} \) is an antiprime algebra (henceforth called a P-algebra) if no nonzero homomorphic image of \( \mathcal{J} \) is strongly semiprime (i.e. every homomorphic image of \( \mathcal{J} \) contains an absolute zero divisor).

**Proposition.** \( P \) is a radical property.

**Proof.**
(a) Every homomorphic image \( \rho(\mathcal{J}) \) of a P-algebra \( \mathcal{J} \) is again a P-algebra for if not, then for some homomorphism \( \delta \), \( \delta(\rho(\mathcal{J})) = \rho(\delta(\mathcal{J})) \) is strongly semiprime and nonzero.

(b) For \( \mathcal{J} \) an ideal of \( \mathcal{J} \) and for \( \mathcal{J} \) and \( \mathcal{J}/\mathcal{J} \) P-algebra, \( \mathcal{J} \) is also a P-algebra for if not, then some nonzero homomorphic image \( \mathcal{J}/\mathcal{J} (R \subset \mathcal{J}) \) is strongly semiprime.

**Case 1.** If \( \mathcal{J} = \mathcal{J} \), then \( \mathcal{J}/\mathcal{J} \) is a strongly semiprime nonzero homomorphic image (projection) of \( \mathcal{J}/\mathcal{J} \) which is assumed to be a P-algebra, a contradiction.

**Case 2.** If \( \mathcal{J} \subset \mathcal{J} \), \( \mathcal{J}/(\mathcal{J} \cap \mathcal{J}) \) is a nonzero homomorphic image of the P-algebra \( \mathcal{B} \). So there is an element \( x \in \mathcal{B} \), \( x \notin \mathcal{J} \), such that \( U_x \mathcal{B} \subset \mathcal{J} \). If \( U_x \mathcal{J} \subset \mathcal{J} \), then \( x (\notin \mathcal{J}) \) is a nonzero absolute zero divisor in \( \mathcal{J}/\mathcal{J} \) which is assumed to be strongly semiprime: contradiction. If \( U_x \mathcal{J} \notin \mathcal{J} \), examine \( U_{U_x(1)} \mathcal{J} \) (where \( U_{U_x(1)} \notin \mathcal{J} \)). \( U_{U_x(1)} \mathcal{J} = U_x U_x \mathcal{J} \subset U_x \mathcal{J} \subset \mathcal{J} \). So \( U_x j \) is a nonzero absolute zero divisor of \( \mathcal{J}/\mathcal{J} \), a contradiction. Hence \( \mathcal{J} \) must be a P-algebra.

(c) Every P-algebra \( \mathcal{J} \) contains a P-ideal \( \mathcal{R} \) which contains every other P-ideal of \( \mathcal{J} \). Let \( \mathcal{R} = \sum \mathcal{B} \) be the sum of all the P-ideals of \( \mathcal{J} \). We claim
\( R \) is a \( P \)-ideal, for take any nonzero homomorphic image \( \rho(R) \) of \( R \): 
\( \rho(R) = \sum \rho(B_i) \). We must find an absolute zero divisor in \( \rho(R) \). Since \( \rho(R) \neq 0 \), \( \rho(B_i) \neq 0 \) for some \( i \); and since \( B_i \) is a \( P \)-ideal, there exists a nonzero element \( z \in \rho(B_i) \) such that \( U_z \rho(B_i) = 0 \). If \( U_z \rho(R) = 0 \), \( z \) is an absolute zero divisor in \( \rho(R) \). So assume \( U_z \rho(R) \neq 0 \) and examine \( U_z \rho(R) \) for some \( i \). \( U_z \rho(R) = U_z U_i \rho(R) \neq 0 \). Hence \( z \) is an absolute zero divisor of \( \rho(R) \). \( R \) is therefore a \( P \)-algebra.

It is easy to see that \( P \) is a radical property of type A because the ideal spanned by all absolute zero divisors is clearly a \( P \)-ideal.

In the case of associative algebras one has the following theorem: For \( R \) an ideal of an associative algebra \( A \), \( R(A) = R(R(A)) \) where \( R \) is the Jacobson radical. In [4] McCrimmon proves this theorem for quadratic Jordan algebras. We now prove this theorem for all hereditary radical properties of type A.

2. The proof.

**Lemma.** For \( B \) an ideal of a quadratic Jordan algebra \( A \), for any radical property \( R \) of type A, and for \( x \in B \), \( U_x B \subseteq R(B) \Rightarrow x \in R(B) \).

**Proof.** \( U_x B \subseteq R(B) \) implies \( U_x B / R(B) = 0 \), i.e., \( x \) is an absolute zero divisor of \( B = B / R(B) \). But since \( R \) is a radical property of type A, \( R(B)R(B) = 0 \) implies \( B / R(B) \) contains no nonzero absolute zero divisors. Hence \( x = 0 \) or \( x \in R(B) \).

**Theorem 1.** For \( B \) an ideal of a quadratic Jordan algebra \( A \), and for \( R \) any radical property of type A, \( R(B) = B \cap R(A) \).

**Proof.** Since \( R(B) \subseteq B \), it is sufficient to show that \( R(B) \) is an \( R \) ideal of \( A \) and therefore \( R(B) \subseteq R(A) \). That is, we must show

1. \( U_{R(B)} A \subseteq R(B) \),
2. \( U_{R(B)} B \subseteq R(B) \).

**Note.** We may as well assume that \( A \) is unital since any quadratic Jordan algebra \( A \) can be imbedded in a unital quadratic Jordan algebra \( A' = A \oplus A \), and any ideal in \( A \) will be an ideal in \( A' \). To prove (1), let \( z \in U_{R(B)} A \), i.e., let \( z \) be a finite linear combination of elements of the form \( U_{x_i} y_i \) where \( x_i \in R(B) \) and \( y_i \in A \). In view of our lemma, since \( z \in B \) it is sufficient to prove \( U_{x_i} B \subseteq R(B) \). Also since \( U_{x_i} B = \sum U_{x_i} y_i \), it is now clear that we will be done if \( U_{x_i} y_i \subseteq R(B) \) for \( x_i \in R(B) \) and \( y_i \in A \); for then \( U_{x_i} B \subseteq R(B) \) implies \( U_{x_i} y_i \subseteq R(B) \) which in turn implies \( U_{x_i} B = U_{x_i} B \subseteq R(B) \) (since \( R(B) \) is an ideal of \( B \) \( \subseteq U_{x_i} B \subseteq R(B) \).

To prove (2) we now let \( z \in U R(B) \), i.e., let \( z \) be a finite linear combination of elements of the form \( U_{x_i} y_i \) where \( x_i \in A \) and \( y_i \in R(B) \). Again
we are done if \( U_x \mathfrak{B} \subseteq R(\mathfrak{B}) \) for \( x \in \mathfrak{J} \) and \( y \in R(\mathfrak{B}) \). By repeated application of the lemma it is sufficient to show \( U_p \mathfrak{B} \subseteq R(\mathfrak{B}) \) for \( p = U_r b', \ q = U_r b, \ r = U_r y \) where now \( x \in \mathfrak{J}, y \in R(\mathfrak{B}), b \) and \( b' \) are arbitrary elements of \( \mathfrak{B} \). But

\[
U_x \mathfrak{B} = U_{U(y)} \mathfrak{B} = U_x U_y U_x \mathfrak{B} = U_{U(x)} U_y U_{U(y)} \mathfrak{B} = (U_x U_y U_x)(U_x U_y U_x)(U_x U_y U_x) (U_x U_y U_x) \mathfrak{B} \quad \text{(by UQJ III)}.
\]

So

\[
U_x \mathfrak{B} \subseteq (U_{U(x)})(U_x U_y)(U_{U(y)}) U_y \mathfrak{B} \quad \text{(since \( \mathfrak{B} \) is an ideal of \( \mathfrak{J} \))} \]
\[
\subseteq (U_{U(x)})(U_x U_y)(U_{U(y)}) R(\mathfrak{B}) \quad \text{(since \( y \in R(\mathfrak{B}) \), an ideal of \( \mathfrak{B} \))} \]
\[
\subseteq (U_{U(x)})(U_x U_y) R(\mathfrak{B}) \]
\[
\subseteq U_{U(y)} U_x U_y - U_{U(x)} U_x U_y R(\mathfrak{B}) \quad \text{(by a linearization of UQJ III)} \]
\[
\subseteq U_{U(y)} U_x U_y - U_{U(\mathfrak{B})} U_x U_y \quad \text{(by \( \mathfrak{B} \) an ideal of \( \mathfrak{J} \))} \]
\[
\subseteq U_{U(y)} U_x U_y \quad \text{R(\(\mathfrak{B}) \subseteq R(\mathfrak{B}).} \]

**Theorem 2.** If \( R \) is any hereditary radical property of type A in a quadratic Jordan algebra \( \mathfrak{J} \) and if \( \mathfrak{B} \) is an ideal of \( \mathfrak{J} \), then \( R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J}). \)

**Proof.** By Theorem 1 it is sufficient to show \( R(\mathfrak{B}) \supseteq \mathfrak{B} \cap R(\mathfrak{J}) \). But since \( R \) is a hereditary radical property, any ideal of an \( R \) ring is again an \( R \) ring. In particular, \( \mathfrak{B} \cap R(\mathfrak{J}) \) is an ideal of \( R(\mathfrak{J}) \) and is therefore an \( R \) ring and an ideal in \( \mathfrak{B} \). Therefore \( \mathfrak{B} \cap R(\mathfrak{J}) \subseteq R(\mathfrak{B}). \)

**Corollary.** For \( R \) nil or quasi-invertible, and for \( \mathfrak{B} \) an ideal of \( \mathfrak{J} \), \( R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J}). \) So, \( \mathfrak{J} \) R-semisimple \( \Rightarrow \mathfrak{B} \) R-semisimple.

**Proof.** It has been mentioned that nil and quasi-invertibility are radical properties of type A. It remains to show that they are hereditary radical properties. But it is clear that every ideal of a nil ring is again nil. Furthermore, the explicit expression for the quasi-inverse of an element \( z \) is

\[
w = U_{U(z)}(z^2 - z) \]

Since

\[
(1 - z)^{-1} = U_{U(z)}(1 - z) = U_{U(z)}(1 - z) - U_{U(z)}(z^2 - z) = 1 - w.
\]

If \( \mathfrak{B} \) is any ideal then the quasi-inverse of any quasi-invertible element of \( \mathfrak{B} \) also belongs to \( \mathfrak{B} \). Quasi-invertibility, therefore, is a hereditary radical property.
REFERENCES


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