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HEREDITARY RADICALS IN JORDAN RINGS

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ABSTRACT. The object of this paper is to examine some radical properties of quadratic Jordan algebras and to show that under certain conditions, $R(\mathfrak{B})=\mathfrak{B}\cap R(\mathfrak{J})$ where $\mathfrak{B}$ is an ideal of a quadratic Jordan algebra $\mathfrak{J}$, $R(\mathfrak{B})$ is the radical of $\mathfrak{B}$, and $R(\mathfrak{J})$ is the radical of $\mathfrak{J}$.

1. **Preliminaries.** We adopt the notation and terminology of an earlier paper [2] concerning quadratic Jordan algebras (defined by the quadratic operators U_x) as opposed to linear Jordan algebras (defined by the linear operators L_x). Thus we have a product $U_x y$ linear in y and quadratic in x satisfying the following axioms as well as their linearizations:

(UQJ I) $U_1=I$ (1 the unit);

(UQJ II) $U_{U(x)y}=U_xU_yU_x$;

(UQJ III) $U_xV_{y,x}=V_{x,y}U_x$ ($V_{x,y}z=\{xyz\}=U_{x,z}y$).

Throughout this paper $\mathfrak{J}$ will denote a quadratic Jordan algebra over an arbitrary ring of scalars Φ.

Define a property R of a class of rings (e.g. associative rings or Jordan rings) to be a *radical property* it it satisfies the following three conditions [1]:

(a) Every homomorphic image of an R ring is again an R ring.

(b) Every ring $\mathfrak{J}$ contains an R ideal $R(\mathfrak{J})$ which contains every other R ideal of $\mathfrak{J}$. The maximal R ideal $R(\mathfrak{J})$ is called the *R radical* of $\mathfrak{J}$.

(c) For $\mathfrak{B}$ an ideal of $\mathfrak{J}$, if $\mathfrak{B}$ and $\mathfrak{J}/\mathfrak{B}$ are R rings, then so is $\mathfrak{J}$.

An immediate consequence of this definition is $R(\mathfrak{B}/R(\mathfrak{B}))=0$. If $R(\mathfrak{B})=0$, $\mathfrak{B}$ is said to be *R semisimple*.

Many well-known radical properties, but not all, satisfy a further condition:

(d) Every ideal of an R ring is again an R ring (i.e. property R is inherited by ideals of an R ring).

If a radical property satisfies condition (d) that property is called *hereditary*.

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In this paper we shall consider all radical properties R of $\mathfrak{J}$ such that if $\mathfrak{J}$ contains no R ideals then $\mathfrak{J}$ contains no absolute zero divisors (where $z \in \mathfrak{J}$ is an absolute zero divisor if $U_z=0$). Henceforth such radical properties will be called *radical properties of type* A.

Two of the more prominent radical properties are quasi-invertibility and nil: An element z belonging to a Jordan algebra $\mathfrak{J}$ is *quasi-invertible* (q.i.) with quasi-inverse $w \in \mathfrak{J}$ if $1-z$ is invertible with inverse $1-w$ in $\Phi 1 \oplus \mathfrak{J}$. A subset is called quasi-invertible if all of its elements are quasi-invertible in $\mathfrak{J}$. The maximal quasi-invertible ideal of an algebra is usually called the *Jacobson radical* and plays an important role in the structure theory. If we define an element $z \in \mathfrak{J}$ to be nilpotent if $z^n=0$ for some n (here powers of an element are defined recursively by $z^0=1$, $z^1=z$, $z^{n+2}=U_z z^n$), then a *nil ideal* is one all of whose elements are nilpotent. There is a unique maximal nil ideal containing all other nil ideals and this is called the nil radical. In [3] McCrimmon shows that nil and quasi-invertibility are radical properties of type A.

A lesser known radical property is that of antiprime: An algebra $\mathfrak{J}$ is *strongly semiprime* if it contains no absolute zero divisors. An algebra $\mathfrak{J}$ is an *antiprime algebra* (henceforth called a P-algebra) if no nonzero homomorphic image of $\mathfrak{J}$ is strongly semiprime (i.e. every homomorphic image of $\mathfrak{J}$ contains an absolute zero divisor).

PROPOSITION. *P is a radical property.*

PROOF. (a) Every homomorphic image $\rho(\mathfrak{J})$ of a P-algebra $\mathfrak{J}$ is again a P-algebra for if not, then for some homomorphism δ, $\delta(\rho(\mathfrak{J}))=\delta\rho(\mathfrak{J})$ is strongly semiprime and nonzero.

(b) For $\mathfrak{B}$ an ideal of $\mathfrak{J}$ and for $\mathfrak{B}$ and $\mathfrak{J}/\mathfrak{B}$ P-algebra, $\mathfrak{J}$ is also a P-algebra for if not, then some nonzero homomorphic image $\mathfrak{J}/\mathfrak{K}$ ($\mathfrak{K} \subset \mathfrak{J}$) is strongly semiprime.

Case 1. If $\mathfrak{B} \subset \mathfrak{K}$, then $\mathfrak{J}/\mathfrak{K}$ is a strongly semiprime nonzero homomorphic image (projection) of $\mathfrak{J}/\mathfrak{B}$ which is assumed to be a P-algebra, a contradiction.

Case 2. If $\mathfrak{B} \nsubseteq \mathfrak{K}$, $\mathfrak{B}/(\mathfrak{B} \cap \mathfrak{K})$ is a nonzero homomorphic image of the P-algebra $\mathfrak{B}$. So there is an element $x \in \mathfrak{B}$, $x \notin \mathfrak{K}$, such that $U_x\mathfrak{B} \subset \mathfrak{K}$. If $U_x\mathfrak{J} \subset \mathfrak{K}$, then x ($\notin \mathfrak{K}$) is a nonzero absolute zero divisor in $\mathfrak{J}/\mathfrak{K}$ which is assumed to be strongly semiprime: contradiction. If $U_x\mathfrak{J} \nsubseteq \mathfrak{K}$, examine $U_{U(x)j}\mathfrak{J}$ (where $U_x j \notin \mathfrak{K}$). $U_{U(x)j}\mathfrak{J}=U_xU_jU_x\mathfrak{J} \subset U_xU_j\mathfrak{B} \subset U_x\mathfrak{B} \subset \mathfrak{K}$. So $U_x j$ is a nonzero absolute zero divisor of $\mathfrak{J}/\mathfrak{K}$, a contradiction. Hence $\mathfrak{J}$ must be a P-algebra.

(c) Every P-algebra $\mathfrak{J}$ contains a P-ideal $\mathfrak{R}$ which contains every other P-ideal of $\mathfrak{J}$: Let $\mathfrak{R}=\sum \mathfrak{B}_\alpha$ be the sum of all the P-ideals of $\mathfrak{J}$. We claim

$\mathfrak{R}$ is a P-ideal, for take any nonzero homomorphic image $\rho(\mathfrak{R})$ of $\mathfrak{R}$: $\rho(\mathfrak{R})=\sum \rho(\mathfrak{B}_\alpha)$. We must find an absolute zero divisor in $\rho(\mathfrak{R})$. Since $\rho(\mathfrak{R})\neq 0$, $\rho(\mathfrak{B}_\alpha)\neq 0$ for some α; and since $\mathfrak{B}_\alpha$ is a P-ideal, there exists a nonzero element $z\in\rho(\mathfrak{B}_\alpha)$ such that $U_z\rho(\mathfrak{B}_\alpha)=0$. If $U_z\rho(\mathfrak{R})=0$, z is an absolute zero divisor in $\rho(\mathfrak{R})$. So assume $U_z\rho(\mathfrak{R})\neq 0$ and examine $U_x\rho(\mathfrak{R})$ ($x=U_zy$, $y\in\rho(\mathfrak{R})$, $x\neq 0$). $U_x\rho(\mathfrak{R})=U_zU_yU_z\rho(\mathfrak{R})\in U_z\rho(\mathfrak{B}_\alpha)=0$. Hence x is an absolute zero divisor of $\rho(\mathfrak{R})$. $\mathfrak{R}$ is therefore a P-algebra.

It is easy to see that P is a radical property of type A because the ideal spanned by all absolute zero divisors is clearly a P-ideal.

In the case of associative algebras one has the following theorem: For $\mathfrak{K}$ an ideal of an associative algebra $\mathfrak{A}$, $R(\mathfrak{K})=\mathfrak{K}\cap R(\mathfrak{A})$ where R is the Jacobson radical. In [**4**] McCrimmon proves this theorem for quadratic Jordan algebras. We now prove this theorem for all hereditary radical properties of type A.

2. The proof.

LEMMA. *For $\mathfrak{B}$ an ideal of a quadratic Jordan algebra $\mathfrak{J}$, for any radical property R of type* A, *and for $x\in\mathfrak{B}$, $U_x\mathfrak{B}\subset R(\mathfrak{B})\Rightarrow x\in R(\mathfrak{B})$.*

PROOF. $U_x\mathfrak{B}\subset R(\mathfrak{B})$ implies $U_{\bar{x}}(\mathfrak{B}/R(\mathfrak{B}))=0$, i.e., $\bar{x}$ is an absolute zero divisor of $\bar{\mathfrak{B}}=\mathfrak{B}/R(\mathfrak{B})$. But since R is a radical property of type A, $R(\mathfrak{B}/R(\mathfrak{B}))=0$ implies $\mathfrak{B}/R(\mathfrak{B})$ contains no nonzero absolute zero divisors. Hence $\bar{x}=\bar{0}$ or $x\in R(\mathfrak{B})$.

THEOREM 1. *For $\mathfrak{B}$ an ideal of a quadratic Jordan algebra $\mathfrak{J}$, and for R any radical property of type* A, *$R(\mathfrak{B})\subset\mathfrak{B}\cap R(\mathfrak{J})$.*

PROOF. Since $R(\mathfrak{B})\subset\mathfrak{B}$, it is sufficient to show that $R(\mathfrak{B})$ is an R ideal of $\mathfrak{J}$ and therefore $R(\mathfrak{B})\subset R(\mathfrak{J})$. That is, we must show

(1) $U_{R(\mathfrak{B})}\mathfrak{J}\subset R(\mathfrak{B})$,

(2) $U_{\mathfrak{J}}R(\mathfrak{B})\subset R(\mathfrak{B})$.

Note. We may as well assume that $\mathfrak{J}$ is unital since any quadratic Jordan algebra $\mathfrak{J}$ can be imbedded in a unital quadratic Jordan algebra $\mathfrak{J}'=\Phi 1\oplus\mathfrak{J}$, and any ideal in $\mathfrak{J}$ will be an ideal in $\mathfrak{J}'$. To prove (1), let $z\in U_{R(\mathfrak{B})}\mathfrak{J}$, i.e., let z be a finite linear combination of elements of the form $U_{x_i}y_i$ where $x_i\in R(\mathfrak{B})$ and $y_i\in\mathfrak{J}$. In view of our lemma, since $z\in\mathfrak{B}$ it is sufficient to prove $U_z\mathfrak{B}\subset R(\mathfrak{B})$. Also since $U_z=U_{\Sigma U(x_i)y_i}=\sum U_{U(x_i)y_i}+\sum_{i<j}U_{U(x_i)y_i,U(x_j)y_j}$, it is now clear that we will be done if $U_{U(x)y}\mathfrak{B}\subset R(\mathfrak{B})$ for $x\in R(\mathfrak{B})$ and $y\in\mathfrak{J}$; for then $U_{U(x_i)y_i}\mathfrak{B}\subset R(\mathfrak{B})$ implies $U_{x_iy_i}\in R(\mathfrak{B})$ which in turn implies $U_{U(x_i)y_i,U(x_j)y_j}\mathfrak{B}\subset R(\mathfrak{B})$. But $U_{U(x)y}\mathfrak{B}=U_xU_yU_x\mathfrak{B}\subset U_xU_yR(\mathfrak{B})$ (since $R(\mathfrak{B})$ is an ideal of $\mathfrak{B}$)$\subset U_x\mathfrak{B}\subset R(\mathfrak{B})$.

To prove (2) we now let $z\in U_{\mathfrak{J}}R(\mathfrak{B})$, i.e., let z be a finite linear combination of elements of the form $U_{x_i}y_i$ where $x_i\in\mathfrak{J}$ and $y_i\in R(\mathfrak{B})$. Again

we are done if $U_{U(x)y}\mathfrak{B} \subset R(\mathfrak{B})$ for $x \in \mathfrak{J}$ and $y \in R(\mathfrak{B})$. By repeated application of the lemma it is sufficient to show $U_p\mathfrak{B} \subset R(\mathfrak{B})$ for $p = U_q b'$, $q = U_r b$, $r = U_x y$ where now $x \in \mathfrak{J}$, $y \in R(\mathfrak{B})$, b and b' are arbitrary elements of $\mathfrak{B}$. But

$$\begin{aligned} U_p\mathfrak{B} &= U_{U(q)b'}\mathfrak{B} = U_q U_{b'} U_q \mathfrak{B} = U_{U(r)b} U_{b'} U_{U(r)b}\mathfrak{B} \\ &= U_r U_b U_r U_{b'} U_r U_b U_r \mathfrak{B} = U_{U(x)y} U_b U_{U(x)y} U_{b'} U_{U(x)y} U_b U_{U(x)y}\mathfrak{B} \\ &= (U_x U_y U_x)(U_b U_x U_y)(U_x U_{b'} U_x)(U_y U_x U_b U_x U_y U_x)\mathfrak{B} \quad \text{(by UQJ III).} \end{aligned}$$

So

$$\begin{aligned} U_p\mathfrak{B} &\subset (U_{U(x)y})(U_b U_x U_y)(U_{U(x)b'})U_y\mathfrak{B} \quad \text{(since } \mathfrak{B} \text{ is an ideal of } \mathfrak{J}) \\ &\subset (U_{\mathfrak{B}})(U_b U_x U_y)(U_{\mathfrak{B}})R(\mathfrak{B}) \quad \text{(since } y \in R(\mathfrak{B}), \text{ an ideal of } \mathfrak{B}) \\ &\subset (U_{\mathfrak{B}})(U_b U_x U_y)R(\mathfrak{B}) \\ &\subset U_{\mathfrak{B}}(U_{U(b,y)x} + U_{U(b)x,U(y)x} - U_y U_x U_b - U_{b,y} U_x U_{b,y})R(\mathfrak{B}) \\ &\qquad \text{(by a linearization of UQJ III)} \\ &\subset U_{\mathfrak{B}}(U_{\mathfrak{B}} + U_{\mathfrak{B},\mathfrak{B}} - U_{R(\mathfrak{B})} U_{\mathfrak{J}} U_{(\mathfrak{B})} - U_{\mathfrak{B},R(\mathfrak{B})} U_{\mathfrak{J}} U_{\mathfrak{B},R(\mathfrak{B})})R(\mathfrak{B}) \\ &\subset U_{\mathfrak{B}} R(\mathfrak{B}) \subset R(\mathfrak{B}). \end{aligned}$$

THEOREM 2. *If R is any hereditary radical property of type* A *in a quadratic Jordan algebra $\mathfrak{J}$ and if $\mathfrak{B}$ is an ideal of $\mathfrak{J}$, then $R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J})$.*

PROOF. By Theorem 1 it is sufficient to show $R(\mathfrak{B}) \supset \mathfrak{B} \cap R(\mathfrak{J})$. But since R is a hereditary radical property, any ideal of an R ring is again an R ring. In particular, $\mathfrak{B} \cap R(\mathfrak{J})$ is an ideal of $R(\mathfrak{J})$ and is therefore an R ring and an ideal in $\mathfrak{B}$. Therefore $\mathfrak{B} \cap R(\mathfrak{J}) \subset R(\mathfrak{B})$.

COROLLARY. *For R nil or quasi-invertible, and for $\mathfrak{B}$ an ideal of $\mathfrak{J}$, $R(\mathfrak{B}) = \mathfrak{B} \cap R(\mathfrak{J})$. So, $\mathfrak{J}$ R-semisimple$\Rightarrow\mathfrak{B}$ R-semisimple.*

PROOF. It has been mentioned that nil and quasi-invertibility are radical properties of type A. It remains to show that they are hereditary radical properties. But it is clear that every ideal of a nil ring is again nil. Furthermore, the explicit expression for the quasi-inverse of an element z is $w = U^{-1}_{(1-z)}(z^2 - z)$ since

$$(1 - z)^{-1} = U^{-1}_{(1-z)}(1 - z) = U^{-1}_{(1-z)}(1 - z)^2 - U^{-1}_{(1-z)}(z^2 - z) = 1 - w.$$

If $\mathfrak{B}$ is any ideal then the quasi-inverse of any quasi-invertible element of $\mathfrak{B}$ also belongs to $\mathfrak{B}$. Quasi-invertibility, therefore, is a hereditary radical property.

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