PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

COALGEBRAS, SHEAVES, AND COHOMOLOGY

D. H. VAN OSDOL[1]

ABSTRACT. The category of sheaves on the topological space X with values in the algebraic category $\mathscr{A}$ is shown to be cotripleable under the stalk category $\mathscr{A}^{|X|}$. If K is a field then the category of K-coalgebras is shown to be cotripleable under the category of K-vector spaces. This makes possible the interpretation of the first group of the associated triple cohomology complex. In particular, for coalgebras our H^1 is isomorphic to Jonah's H^2.

1. **Introduction.** The theory of triples T and their associated T-algebras has been a unifying force in mathematics as well as a rapidly expanding area of research ([1], [3], [6], [8], [9]). The dual theory of cotriples G and their associated G-coalgebras has been relatively neglected (however, see the article of Applegate and Tierney in [3]). We suggest that this neglect is unwarranted, and prove in this note that two important categories are categories of (cotriple-theoretic) coalgebras.

First, the stalk functor from algebraic sheaves on the topological space X to sheaves on the space X made discrete is cotripleable. Second, if K is a field then the forgetful functor from K-coalgebras to K-vector spaces is cotripleable. We enunciate the dual of Beck's classification theorem [1] and use it to give an interpretation of the cohomology for sheaves and coalgebras.

Throughout this paper, if A and B are objects of a category $\mathscr{A}$ then (A, B) will denote the set of $\mathscr{A}$-morphisms from A to B; whereas if $\mathscr{A}$ is a small category and $\mathscr{B}$ is a category then $\mathscr{B}^{\mathscr{A}}$ will denote the category whose objects are covariant functors $\mathscr{A}\to\mathscr{B}$ and whose morphisms are natural transformations of functors. A symbol X will denote either the object X or the identity morphism on X.

2. **Sheaves of algebras.** Let $\mathscr{A}$ be an algebraic category [9]. Among the algebraic categories are all the usual categories of algebras, for example

Presented to the Society, September 3, 1971; received by the editors September 27, 1971.

AMS 1970 *subject classifications*. Primary 18C15, 18F20, 18G10, 18H40; Secondary 55B30, 18D35.

Key words and phrases. Sheaves of algebras, cotripleable, coalgebras, principal objects, cogroup object, extension, Jonah cohomology.

[1] Work supported by National Science Foundation grant GP-29067.

the categories of groups, R-modules, rings, Lie rings, algebras, and Lie algebras. Let X be a topological space and let $\mathscr{P}(X, \mathscr{A})$ (respectively $\mathscr{F}(X, \mathscr{A})$) be the category of presheaves (respectively sheaves) on X with values in $\mathscr{A}$ [7]. Let $|X|$ be the discrete category on the set X.

The stalk functor $S:\mathscr{P}(X, \mathscr{A})\to\mathscr{A}^{|X|}$ is defined by $(SP)\{x\}=P_x=\operatorname{colim} PV$ where the colimit is taken over all open subsets V of X which contain x; note that this is a directed colimit. The right adjoint to S, $Q:\mathscr{A}^{|X|}\to\mathscr{P}(X, \mathscr{A})$, is defined as follows: given $\{A_x\}$ in $\mathscr{A}^{|X|}$ and an open subset V of X, $(Q\{A_x\})V=\prod A_x$ where the product is taken over all points x in V. The definitions of S and Q on morphisms are obvious. Actually since $Q\{A_x\}$ is a sheaf [7], we have the "same" adjoint pair $\mathscr{F}(X, \mathscr{A})\to\mathscr{A}^{|X|}\to\mathscr{F}(X, \mathscr{A})$. Note that QS is the Godement standard construction [**4**].

THEOREM 1. *$S:\mathscr{F}(X, \mathscr{A})\to\mathscr{A}^{|X|}$ is crudely cotripleable, and $\mathscr{F}(X, \mathscr{A})$ is cocomplete.*

PROOF. Since $\mathscr{A}$ is complete, so is $\mathscr{F}(X, \mathscr{A})$. For sheaves in an algebraic category, a morphism of sheaves is an isomorphism if and only if it is an isomorphism on each stalk [**4**]. In an algebraic category directed colimits commute with finite limits [**9**]. Hence we have verified the conditions of the CCT [**1**].

Thus there exists an equivalence $\Psi':(\mathscr{A}^{|X|})_G\to\mathscr{F}(X, \mathscr{A})$. Its composition with the comparison functor $\Psi:\mathscr{P}(X, \mathscr{A})\to(\mathscr{A}^{|X|})_G$ is easily described: $\Psi'\Psi=\text{equalizer }(\eta QS, QS\eta)$. One can then use Theorem 1 to prove the following:

COROLLARY 2. *$\Psi'\Psi$ is the associated sheaf functor, that is, $\Psi'\Psi\dashv i$ where $i:\mathscr{F}(X, \mathscr{A})\to\mathscr{P}(X, \mathscr{A})$ is the inclusion functor.*

For interpretation of these results, see [**11**].

3. **Coalgebras.** Let K be a field and let $\mathscr{M}$ be the category of K-modules (vector spaces). A K-coalgebra is a three-tuple (C, Δ, c) where C is a K-module and $\Delta:C\to C\otimes C$, $c:C\to K$ are K-module homomorphisms such that $(c\otimes C)\cdot\Delta=C=(C\otimes c)\cdot\Delta$ and $(\Delta\otimes C)\cdot\Delta=(C\otimes\Delta)\cdot\Delta$. A morphism of K-coalgebras, $f:(C, \Delta, c)\to(C', \Delta', c')$, is a K-module homomorphism $f:C\to C'$ such that $\Delta'\cdot f=(f\otimes f)\cdot\Delta$ and $c'\cdot f=c$. Let $\mathscr{C}$ be the category of K-coalgebras and $S:\mathscr{C}\to\mathscr{M}$ the obvious forgetful functor.

THEOREM 3. *$S:\mathscr{C}\to\mathscr{M}$ is cotripleable.*

PROOF. It is proved in [**10**] that S has a right adjoint. Clearly S reflects isomorphisms. Given morphisms

$$C_1 \rightrightarrows C_2$$

such that

$$E \xrightarrow{i} SC_1 \rightrightarrows SC_2$$

is an absolute equalizer, we give E a coalgebra structure in the following way. Since $F(M)=M\otimes M$, $F(f)=f\otimes f$ defines a functor $\mathscr{F}:\mathscr{M}\to\mathscr{M}$, it follows that

$$E \otimes E \to SC_1 \otimes SC_1 \rightrightarrows SC_2 \otimes SC_2$$

is an equalizer. Hence there is a unique $\Delta:E\to E\otimes E$ such that $(i\otimes i)\cdot\Delta=\Delta_1\cdot i$. If we let $e=c_1\cdot i$, then it is easy to verify that (E, Δ, e) is a coalgebra and

$$E \to C_1 \rightrightarrows C_2$$

is the equalizer in $\mathscr{C}$. Hence we have verified the conditions of the PCT [**8**].

4. **Beck's classification theorem.** Let $S:\mathscr{A}\to\mathscr{B}$ be cotripleable with right adjoint $Q:\mathscr{B}\to\mathscr{A}$. For each object A of $\mathscr{A}$ we get a semisimplicial complex T^*A, where in dimension $n\geqq 0$ we have $(QS)^{n+1}A$ and for each $i\leqq n+1$, $\eta_i=(QS)^i\eta(QS)^{n+1-i}A:(QS)^{n+1}A\to(QS)^{n+2}A$. Suppose $\mathscr{B}$ is finitely cocomplete (which implies that $\mathscr{A}$ is finitely cocomplete) and let $C=(C, \gamma:C\to C\coprod C,\ c:C\to 0,\ l:C\to C)$ be an abelian cogroup object in $\mathscr{A}$. Then (C, T^*A) is a cochain complex, where $d:(C, T^{n+1}A)\to(C, T^{n+2}A)$ is defined by $d(f)=\sum_{i=0}^{n+1}(-1)^i\eta_i\cdot f$. We define $H^n(C, A)=$ the nth homology group of (C, T^*A).

THEOREM 4. $H^0(C, A)\cong(C, A)$.

PROOF. See Beck [**1**].

To interpret $H^1(C, A)$ we need some more definitions. A pair (D, ρ) is a C-object if D is an object of $\mathscr{A}$, $\rho:D\to D\coprod C$ in $\mathscr{A}$, $(D\coprod\gamma)\cdot\rho=(\rho\coprod C)\cdot\rho$, and $(D\coprod c)\cdot\rho=D$. A C-principal object under A is a morphism $i:A\to D$ in $\mathscr{A}$ where (D, ρ) is a C-object and:

(i) $(i\coprod C)\cdot i_1=\rho\cdot i$ where $i_1:A\to A\coprod C$ is the injection.

(ii) For each pair $f_1, f_2:D\to X$ in $\mathscr{A}$ such that $f_1\cdot i=f_2\cdot i$ there exists a unique $g:C\to X$ in $\mathscr{A}$ such that $(f_1\coprod g)\cdot\rho=f_2$.

(iii) There is given as part of the structure a map $s:SD\to SA$ such that $s\cdot Si=SA$.

A morphism $(D, \rho)\to(D', \rho')$ of C-principal objects under A is an $\mathscr{A}$-morphism $f:D\to D'$ such that $f\cdot i=i'$ and $(f\coprod C)\cdot\rho=\rho'\cdot f$. Let $\mathscr{P}\mathscr{O}$ be the category of C-principal objects under A and their morphisms.

THEOREM 5. *$H^1(C, A)$ is isomorphic to the set of connected components of $\mathscr{P}\mathscr{O}$.*

PROOF. See Beck [**1**].

5. **Specific examples.** We now apply Theorem 5 to the situations studied in §§2 and 3.

EXAMPLE 1. If $\mathscr{A}$ is the category of abelian groups and homomorphisms then every object C in $\mathscr{F}(X, \mathscr{A})$ is an abelian cogroup where $\gamma: C \to C \coprod C$ is the diagonal map, $c: C \to 0$ is the zero map, and $l: C \to C$ is inversion. If A is another sheaf of abelian groups then $H^1(C, A)$ will be in one-one correspondence with the set of equivalence classes of short exact sequences $O \to A \to D \to C \to O$ of sheaves of abelian groups such that the sequence of abelian groups $0 \to A_x \to D_x \to C_x \to 0$ is split (as abelian groups) for each x in X. Of course two such sequences are equivalent if there exists a commutative diagram

$$\begin{tikzcd}
& & D \arrow[dd] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\
& & D' \arrow[ur] & &
\end{tikzcd}$$

of sheaves of abelian groups. When we take $C = Z =$ the constant sheaf of integers, we conclude that $(Z, T^*A) \cong \Gamma T^*A$. Hence $H^n(Z, A) \cong H^n(X, A)$, the usual cohomology of X with coefficients in A (since T^*A is the Godement resolution of A) [**4**]. On the other hand, the sequence $0 \to A_x \to D_x \to Z_x \to 0$ will automatically split because Z_x is free. Hence $H^1(X, A) \cong H^1(Z, A) \cong \mathrm{Ext}^1(Z, A)$. Similarly, if C has projective stalks then $H^1(C, A) \cong \mathrm{Ext}^1(C, A)$.

EXAMPLE 2. One can define $H^0(C, A)$ and $H^1(C, A)$, essentially as above, when C is a not-necessarily-abelian cogroup (see [**1**]). If $\mathscr{A}$ is the category of commutative K-algebras, then the cogroups in $\mathscr{A}$ are the affine algebraic groups [**2**]. Let C be an object in $\mathscr{F}(X, \mathscr{A})$ which is a sheaf of affine algebraic groups: then C is a cogroup object in $\mathscr{F}(X, \mathscr{A})$. Let A be a sheaf of commutative K-algebras. Then $H^1(C, A)$ is in one-one correspondence with equivalence classes of sheaves of K-algebras D such that D contains A as a subsheaf and $D_x \cong A_x \otimes C_x$ for each x in X. In other words, $H^1(C, A)$ tells how many inequivalent sheaves there are which have stalks $A_x \otimes C_x$ and contain A as a subsheaf.

REMARK. Example 2 is typical for sheaves with values in some algebraic category $\mathscr{A}$. Roughly speaking, $H^1(C, A)$ will classify sheaves containing A and having stalks $A_x \coprod C_x$.

EXAMPLE 3. Let $\mathscr{C}$ be the category of K-coalgebras defined in §3, and let C be a coalgebra. The comma category $(C, \mathscr{C})$ has objects $C \xrightarrow{i} \Gamma$ in C and maps $C \xrightarrow{i} \Gamma \xrightarrow{f} \Gamma' = C \xrightarrow{i'} \Gamma'$. One can show that $(C, S): (C, \mathscr{C}) \to (SC, \mathscr{M})$ is cotripleable, where $(C, S)(C \to \Gamma) = SC \to S\Gamma$ and similarly for maps. The dual situation is considered by Beck in [**1**]. Let $\mathrm{Coab}(C, \mathscr{C})$ be the category of abelian cogroups in $(C, \mathscr{C})$; we claim that $\mathrm{Coab}(C, \mathscr{C})$ is equivalent to the category of C-C-comodules. Given an abelian cogroup $i: C \rightarrow \Gamma$, we first note that the cogroup counit $d: \Gamma \rightarrow C$ yields an isomorphism of K-modules $\Gamma \cong C \oplus M$ where M is the cokernel of $i: C \rightarrow \Gamma$. Replacing Γ by $C \oplus M$, $i: C \rightarrow C \oplus M$ becomes the injection into the first factor, $i(x) = (x, 0)$. Using the matrix notation of Jonah **[5]**, the coalgebra comultiplication $\gamma: C \oplus M \rightarrow (C \oplus M) \otimes (C \oplus M)$ can be represented by

$$\gamma = \begin{bmatrix} \Delta & h \\ 0 & m_l \\ 0 & m_r \\ 0 & m \end{bmatrix}$$

where $\Delta: C \rightarrow C \otimes C$ is the coalgebra costructure, $h: M \rightarrow C \otimes C$, $m_l: M \rightarrow C \otimes M$, $m_r: M \rightarrow M \otimes C$, and $m: M \rightarrow M \otimes M$. Since $d = [C\,0]$ is a map of coalgebras, $(d \otimes d) \cdot \gamma = \Delta \cdot d$ implies $h = 0$. Also if $c: C \rightarrow K$ is the coalgebra counit and $c': C \oplus M \rightarrow K$ is the coalgebra counit then $d \cdot c = c'$ implies $c' = [c\,0]$.

We are next interested in the cogroup comultiplication. Notice that the coproduct of $i: C \rightarrow C \oplus M$ with itself is $C \rightarrow C \oplus M \oplus M$, and hence the cogroup comultiplication will be a coalgebra map $e: C \oplus M \rightarrow C \oplus M \oplus M$. One checks that the coalgebra costructure $\delta: C \oplus M \oplus M \rightarrow (C \oplus M \oplus M) \otimes (C \oplus M \oplus M)$ is represented by

$$\delta = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & m_l & 0 \\ 0 & 0 & m_l \\ 0 & m_r & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_r \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix}.$$

Since

$$e = \begin{bmatrix} C & 0 \\ 0 & M \\ 0 & M \end{bmatrix}$$

is a coalgebra map, $(e\otimes e)\cdot\gamma=\delta\cdot e$ implies that $m=0$. Hence

$$\gamma = \begin{bmatrix} \Delta & 0 \\ 0 & m_l \\ 0 & m_r \\ 0 & 0 \end{bmatrix}.$$

Now $[(C\oplus M)\otimes\gamma]\cdot\gamma=[\gamma\otimes(C\oplus M)]\cdot\gamma$ says that $(\Delta\otimes M)\cdot m_l=(C\otimes m_l)\cdot m_l$, $(M\otimes\Delta)\cdot m_r=(m_r\otimes C)\cdot m_r$, and $(C\otimes m_r)\cdot m_l=(m_l\otimes C)\cdot m_r$. Moreover, $[c'\otimes(C\oplus M)]\cdot\gamma=C\oplus M$ implies $(c\otimes M)\cdot m_l=M$ and $[(C\oplus M)\otimes c']\cdot\gamma=C\oplus M$ implies $(M\otimes c)\cdot m_r=M$. Such a K-module (M, m_l, m_r) is precisely a two-sided C-comodule, and hence Coker: Coab$(C, \mathscr{C})\to C$-C-comodules defines a functor.

On the other hand, given a two-sided C-comodule (M, m_l, m_r) we define $\Gamma=C\oplus M$ as a K-module. Let

$$\gamma = \begin{bmatrix} \Delta & 0 \\ 0 & m_l \\ 0 & m_r \\ 0 & 0 \end{bmatrix}$$

and $c'=[c\ 0]$, thus making Γ into a coalgebra and $i=[{C \atop 0}]$ into a coalgebra map. Finally

$$e = \begin{bmatrix} C & 0 \\ 0 & M \\ 0 & M \end{bmatrix}, \qquad d = [C \quad 0], \qquad l = \begin{bmatrix} C & 0 \\ 0 & -M \end{bmatrix}$$

give $i: C\to\Gamma$ an abelian cogroup structure, and one verifies that this assignment is the functorial inverse to Coker. Using this category equivalence it is possible to reinterpret the group of morphisms $(i: C\to\Gamma, i': C\to\Gamma')$ where i is an abelian cogroup in $(C, \mathscr{C})$. Given a C-C-comodule M and coalgebra (Γ', γ', c'), a K-module morphism $f: M\to\Gamma'$ is an i' coderivation from M to Γ' if $\gamma'\cdot f=(i'\otimes f)\cdot m_l+(f\otimes i')\cdot m_r$ and $c'\cdot f=0$. Then (i, i') is in one-one correspondence with the group of all i' coderivations from M to Γ', where $M=\text{Coker } i$.

Applying Theorem 5 to this situation, we let $i: C\to\Gamma$ be an abelian cogroup in $(C, \mathscr{C})$ and interpret $H^1(i, C)$ where C is the initial object in $(C, \mathscr{C})$. An i-principal object under C will be a coalgebra map $j: C\to\Gamma'$ such that Sj is isomorphic to the coproduct of Si and SC, which of course is Si. Hence as K-modules, Γ' and $\Gamma=C\oplus M$ are isomorphic. It follows that if we let $JH^2(M, C)$ be Jonah's second cohomology group [5] then $H^1(i, C)\cong JH^2(M, C)$ where $M=\text{Coker } i$. Note finally that one can compute $H^n(i, C)$ as the nth homology group of the complex Coder (M, T^*C).

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Department of Mathematics, University of New Hampshire, Durham, New Hampshire 03824