A NOTE ON POLYHEDRAL BANACH SPACES

ALAN GLEIT AND ROBERT MCGUIGAN

Abstract. We give a sufficient condition for an infinite-dimensional Banach space X to be polyhedral. If X* is an L-space this condition is also necessary.

1. Introduction. In this paper all Banach spaces will be over the scalar field of real numbers. If X is a Banach space, the unit ball of X is denoted S(X). If K is a convex subset of a vector space, Ext(K) is the set of all extreme points of K. Thus Ext S(X) is the set of all extreme points of the unit ball of X. A Banach space X is polyhedral iff for every finite-dimensional subspace E \subseteq X, S(E) has only finitely many extreme points.

The following lemma is essentially contained in [2]. We include a brief proof for the sake of completeness.

Lemma 1.1. If E is a finite-dimensional Banach space such that S(E) has infinitely many extreme points, then S(E*) has infinitely many extreme points.

Proof. Recall that a point x \in E with \|x\| = 1 is a smooth point of S(E) iff there is a unique functional f \in E* such that \langle f, x \rangle = \|f\| = 1. According to a theorem of Mazur [8], the set of smooth points of S(E) is dense in the boundary of S(E). Let \mathcal{S} \subseteq Ext S(E*) be the set of all functionals of norm one in E* that attain their norms at smooth points of S(E). We have

$$S(E) = \bigcap_{f \in \mathcal{S}} \{x \mid \langle f, x \rangle \leq 1\}.$$ 

If \mathcal{S} were finite, then S(E) would be the intersection of a finite number of half-spaces and hence a polyhedron. This contradiction implies that \mathcal{S} is infinite.

Remark. Klee also shows in [2] that a finite-dimensional Banach space E is polyhedral iff E* is polyhedral. Lindenstrauss has shown [6] that no infinite-dimensional conjugate space can be polyhedral.

Theorem 1.2. For an infinite-dimensional Banach space X consider the
following three properties:

1. There do not exist a weak*-accumulation point \( q \) of \( Ext S(X^*) \) and \( x \in X \) satisfying \( \langle q, x \rangle = \|x\| = \|q\| = 1 \).

2. \( X \) is polyhedral.

3. No subspace of \( X \) is linearly isometric to \( c \).

Then \((1) \Rightarrow (2) \Rightarrow (3)\). When \( X^* \) is an \( L \)-space, then \((1) \) through \((3)\) are equivalent statements.

**Proof.** We establish the contrapositives of the required implications.

\((1) \Rightarrow (2)\). Suppose \( X \) is not polyhedral. Then there is a finite-dimensional subspace \( E \subseteq X \) such that \( S(E) \) has infinitely many extreme points. By the previous lemma, \( S(E^*) \) has infinitely many extreme points. Since \( S(E^*) \) is norm-compact, we can choose a sequence \( (f_n) \) of distinct elements of \( Ext S(E^*) \) such that \( f_n \rightarrow g \in E^* \). Now \( g \) attains its norm on \( S(E) \) (since \( E \) is reflexive) and \( \|g\| = 1 \).

If \( f \in Ext S(E^*) \) then \( \{ f \in S(X^*) \mid \| f \| = 1 \text{ and } \int f = f \} \) is a weak*-closed face of \( S(X^*) \) and hence contains an element of \( Ext S(X^*) \). Thus, for each \( n \) there is some \( f_n \in Ext S(X^*) \) such that \( f_n \vert E = f_n \). By weak*-compactness of \( S(X^*) \), \( (f_n) \) has a weak*-cluster point \( h \in S(X^*) \). Clearly \( h \vert E = g \). Thus \( \|h\| = 1 \) and \( h \) attains its norm.

\((2) \Rightarrow (3)\). Suppose \( c \) is linearly isometric to a subspace of \( X \). Then \( X \) is not polyhedral since \( c \) is not polyhedral.

The proof of \((3) \Rightarrow (1)\) for spaces whose duals are \( L \)-spaces requires some additional lemmas, and is therefore postponed until the next section.

**Remark.** Implication \((1) \Rightarrow (2)\) of the previous theorem generalizes Klee's theorem [3] that \( c_0 \) is polyhedral, since clearly \( c_0 \) satisfies \((1)\).

2. Lindenstrauss spaces. We first recall some terminology. A vector lattice \( V \) is an \( L \)-space if its norm satisfies

\[
\|p + q\| = \|p\| + \|q\|, \quad p, q \in V^+, \quad \text{and} \quad \|p\| = \|p^+\| + \|p^-\|.
\]

A Banach space whose dual space is an \( L \)-space is called a Lindenstrauss space (see [1] for terminology). If a Lindenstrauss space can be ordered in a way compatible with the duality and the natural order in the \( L \)-space, it is called a simplex space.

Suppose that \( W \) is a normed space. We say that \( r \in W \) dominates \( p \), written \( p < r \), if \( r = p + q \) and \( \|r\| = \|p\| + \|q\| \). A nonempty subset \( H \) of \( S(W) \) is a biface if:

- \( B_1 \). \( H \) is convex and symmetric.
- \( B_2 \). If \( x \neq 0 \) is in \( H \) then \( x/\|x\| \) is in \( H \).
- \( B_3 \). If \( q \in H \) and \( p < q \), then \( p \in H \).

It is easily seen that \( Ext(H) \subseteq Ext S(W) \) [1, Lemma 4.5].

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
**Lemma 2.1.** Let $W$ be a normed space and $F \subseteq S(W)$ a convex subset consisting of elements of norm one. If $H = \text{co}(F \cup -F)$ is a biface, then $F$ is a face.

**Proof.** Let $x, y \in S(W)$ and $0 < \lambda < 1$ satisfy $f = \lambda x + (1 - \lambda)y \in F$. Since $1 = \|f\| \leq \lambda \|x\| + (1 - \lambda) \|y\| \leq \lambda + (1 - \lambda) = 1$, we conclude that $\|x\| = \|y\| = 1$, and that $f$ dominates both $\lambda x$ and $(1 - \lambda)y$. From properties B2 and B3 we easily see that $x, y \in H$. Thus there are elements $p_x, p_y, q_x, q_y \in F$ and $0 \leq \alpha, \beta \leq 1$ satisfying

$$x = \alpha p_x - (1 - \alpha)q_x, \quad y = \beta p_y - (1 - \beta)q_y.$$ 

Then

$$f = (\lambda \alpha p_x + (1 - \lambda)\beta p_y) - (\lambda(1 - \alpha)q_x + (1 - \lambda)(1 - \beta)q_y).$$

Let $\mu = \lambda \alpha + (1 - \lambda)\beta$. Then

$$1 - \mu = \lambda(1 - \alpha) + (1 - \lambda)(1 - \beta).$$

Assume $0 < \mu < 1$. Then

$$a = \mu^{-1}(\lambda \alpha p_x + (1 - \lambda)\beta p_y) \in F,$$

$$b = (1 - \mu)^{-1}(\lambda(1 - \alpha)q_x + (1 - \lambda)(1 - \beta)q_y) \in F.$$ 

But then

$$\mu f + (1 - \mu)b = \mu^2 a + (1 - \mu)^2 b \in F.$$ 

Since every element of $F$ has norm one, we get

$$1 = \|\mu^2 a + (1 - \mu)^2 b\| \leq \mu^2 + (1 - \mu)^2$$

which implies $\mu = 0$ or $\mu = 1$. This is a contradiction. Hence $\mu = 0$ or $\mu = 1$.

If $\mu = 0$, then $\alpha = \beta = 0$ and so $f = -q_x - q_y \in F \cap -F$ which is clearly impossible. So $\mu = 1$, and thus $\alpha = \beta = 1$. Hence $x = p_x \in F$ and $y = p_y \in F$.

The following result is due to A. Lazar.

**Proposition 2.2.** Let $H$ be a nonzero biface in an $L$-space $V$. Then there is a face $F \subseteq S(V)$ such that $H = \text{co}(F \cup -F)$.

**Proof.** Let $F = \{p \in H | \|p\| = 1, p = p^+\}$. $F$ is convex since $V$ is an $L$-space. Let $q \in H$, $q \neq 0$. Then $q = q^+ - q^-$. If $q^+ \neq 0$ we have $q^+ / \|q^+\| \in F$. Otherwise $q^- / \|q^-\| \in F$, and in any case, $F \neq \emptyset$. Hence $0 \in \text{co}(F \cup -F)$.

Since $H$ is convex and symmetric, $\text{co}(F \cup -F) \subseteq H$. Conversely, let $q \in H$. Then $q = q^+ - q^-$, and $\|q\| = \|q^+\| + \|q^-\|$. Hence $q > q^+, q > q^-$, and, by B3, $q^+, q^- \in H$. Thus $q^+ / \|q^+\|$ and $-q^- / \|q^-\|$ belong to $F$. The equality

$$\frac{q^+}{\|q\|} = \frac{\|q^+\|}{\|q\|} \left(\frac{q^+}{\|q^+\|}\right) - \frac{\|q^-\|}{\|q\|} \left(\frac{q^-}{\|q^-\|}\right)$$

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
together with \(0 \in \text{co}(F \cup -F)\) clearly yields \(q \in \text{co}(F \cup -F)\). Hence \(H = \text{co}(F \cup -F)\). Now Lemma 2.1 implies that \(F\) is a face and the proof is complete.

For completeness we quote here, in our terminology, the generalization by Lazar and Lindenstrauss of the Edwards separation theorem.

**Theorem 2.3** [5, Theorem 2.1]. Let \(X\) be a Lindenstrauss space. Let \(g : S(X^*) \rightarrow (-\infty, \infty]\) be a concave weak*-lower semicontinuous function satisfying
\[
g(v^*) + g(-v^*) \geq 0, \quad v^* \in S(X^*).
\]
Let \(H\) be a weak*-closed biface of \(S(X^*)\). Assume that \(f\) is a weak*-continuous affine symmetric real-valued function on \(H\) such that \(f \leq g\mid H\). Then there is an element \(v \in X\) which satisfies
1. \(\langle v^*, v \rangle = f(v^*), \quad v^* \in H,\)
2. \(\langle y^*, v \rangle \leq g(y^*), \quad y^* \in S(X^*).\)

**Corollary 2.4.** Let \(X\) be a Lindenstrauss space and \(H\) a weak*-closed biface of \(S(X^*)\). Let \(f \) be a weak*-continuous affine symmetric real-valued function on \(H\). Then there is an element \(v \in X\) which satisfies
1. \(\langle v^*, v \rangle = f(v^*), \quad v^* \in H,\)
2. \(\|v\| = \|f\|_\infty.\)

**Proof.** We may take \(g = \|f\|_\infty\) in Theorem 2.3.

**Lemma 2.5.** Let \(V\) be an \(L\)-space. Let \(P = \{p_\alpha \mid \alpha \in A\}\) be a subset of \(\text{Ext } S(V^*)\) such that \(p_\alpha \not\leq \pm p_\beta\) for all \(\alpha, \beta \in A\). Let \(\{p_n\}\) be a sequence drawn from \(P\). Then if \(\sum |\alpha_n| \leq 1,\)
\[
\sum_{n=1}^{\infty} \alpha_np_n = 0 \Rightarrow x_n = 0, \quad \text{each } n.
\]
In particular, \(P\) is linearly independent.

**Proof.** By changing the sign of \(\alpha_n\) where necessary we may assume \(p_n = p_n^+\) for all \(n\). The result then follows from obvious triangle inequalities.

We say that a subset \(D\) of \(S(X^*)\), \(X\) a Lindenstrauss space, is *symmetrically dilated* if for each \(p \in D\), the extreme points of some weak*-closed biface containing \(p\) are contained in \(D\).

**Corollary 2.6.** Let \(X\) be a Lindenstrauss space. Suppose \(\{p_n\} \subseteq \text{Ext } S(X^*)\) converges weak* to \(q \in \text{co}(x_1, \ldots, x_N)\) for \(x_i \in \text{Ext } S(X^*)\), \(i = 1, \ldots, N\). Suppose \(p_n \not\leq \pm p_m, \not\geq x_i, \not\leq x_i\) and \(x_i \not\leq \pm x_j\) for all \(n, m, i, j\). Then
\[
F = \overline{\text{co}}(\bigcup \{p_n\}, x_1, \ldots, x_N)
\]
is a proper infinite-dimensional weak*-closed face of \(S(X^*).\)
Proof. Let $K=\text{co}(\pm x_1, \ldots, \pm x_N)$. $K$ is clearly the smallest weak*-
closed biface containing $q$ [1, Proposition 4.6]. Let $D=\{\pm p_n\} \cup K$. Then $D$
is symmetrically dilated and compact. Hence the closed convex hull $H$ of
$D$ is a biface of $S(X^*)$ [1, Theorem 5.8]. Clearly
$$H = \overline{\text{co}}(\bigcup \{\pm p_n\}, \pm x_1, \ldots, \pm x_N) = \text{co}(F \cup -F).$$
Also,
$$F = \left\{ \sum \alpha_n p_n + \sum \beta_i x_i \mid \alpha_n \geq 0, \beta_i \geq 0, \sum \alpha_n + \sum \beta_i = 1 \right\},$$
and so
$$H = \left\{ \sum \alpha_n p_n + \sum \beta_i x_i \mid \sum |\alpha_n| + \sum |\beta_i| \leq 1 \right\}.$$

On $H$ we define a weak* -continuous affine symmetric function $f$ by
$$f\left( \sum \alpha_n p_n + \sum \beta_i x_i \right) = \sum \alpha_n + \sum \beta_i.$$

$f$ is well defined by Lemma 2.5. Clearly $f|F \equiv 1$. By Corollary 2.4, there
is a $v \in X$ which extends $f$ and satisfies $\|v\| = \|f\| = 1$. As each $q \in F$
satisfies $\langle q, v \rangle = 1$ we have that $\|q\| = 1$ for each $q \in F$. Hence, by Lemma
2.1, $F$ is a face of $S(X^*)$. $F$ is infinite dimensional since
$$P = \{ \bigcup \{p_n\}, x_1, \ldots, x_N \}$$
is linearly independent by Lemma 2.5.

We are now prepared to complete the proof of Theorem 1.2. First we
recall that if $x$ is a Lindenstrauss space, Lazar [4, Theorem 3] has shown
that $X$ has no subspace linearly isometric to $c$ if $S(X^*)$ has no proper
infinite-dimensional weak* -closed face.

Proof of Theorem 1.2 (Continued). We shall first show that the
negation of (1) implies the existence of a proper infinite-dimensional
weak* -closed face of $S(X^*)$ if $X$ is a separable Lindenstrauss space. Assuming
(1) is false, we can find a weak* -accumulation point $q$ of $\text{Ext} S(X^*)$
and a $v \in X$ satisfying $\langle q, v \rangle = \|v\| = \|q\| = 1$. Let
$$G = \{ p \in S(X^*) \mid p(v) = 1 \}.$$

Then $G$ is a weak* -closed proper nonempty face of $S(X^*)$. If it is infinite
dimensional, we are done. So assume that $G$ is finite dimensional. By
Corollary 2.5 we easily conclude that $G = \text{co}(x_1, \ldots, x_N)$ for some
$x_i \in \text{Ext} S(X^*)$. Since $S(X^*)$ is weak* -metrizable, there is a sequence
$\{p_n\} \subseteq \text{Ext} S(X^*)$ weak* -converging to $q$. Without loss of generality, we
may assume that $p_n \neq \pm p_m, x_i$ for all $n, m, i$. But then Corollary 2.6
provides us with a proper infinite-dimensional weak* -closed face of $S(X^*)$. 
We shall now reduce the nonseparable case to the separable case by showing that if \((1)\) is false for a nonseparable Lindenstrauss space \(X\) then it is false for a separable Lindenstrauss subspace of \(X\). Let \(X\) be nonseparable and let \(q\) be a weak*-accumulation point of \(\text{Ext} S(X^*)\) with \(x_0 \in X\) such that 
\[
\langle q, x_0 \rangle = \|x_0\| = \|q\| = 1.
\]
Choose \(q_1 \in \text{Ext} S(X^*)\) such that \(|1 - \langle q_1, x_0 \rangle| < 2^{-1}\). Let \(x_1 \in X\) satisfy 
\[
\langle q_1, x_1 \rangle = \|x_1\| = 1.
\]
Such an \(x_1\) exists by Corollary 2.4. Define 
\[
F_1 = \{x^* \in S(X^*) : \langle x^*, x_1 \rangle = 1\}.
\]
Now choose \(q_2 \in \text{Ext} S(X^*)\) such that \(|1 - \langle q_2, x_0 \rangle| < 2^{-2}\) and \(q_2 \notin F_1\). By Lazar's theorem [4, Theorem 3], \(F_1\) is finite dimensional. By Corollary 2.4 choose \(x_2 \in X\) such that \(x_2\) vanishes on \(F_1\) and \(\langle q_2, x_2 \rangle = \|x_2\| = 1\)
Inductively we construct two sequences \(\{q_n\} \subseteq \text{Ext} S(X^*), \{x_n\} \subseteq X\) such that 
1. \(|1 - \langle q_n, x_0 \rangle| < 2^{-n}\),
2. \(q_n \notin F_i = \{x^* \in S(X^*) : \langle x^*, x_i \rangle = 1\}\) for \(i < n\),
3. \(\|x_n\| = \langle q_n, x_n \rangle = 1\),
4. \(x_n\) vanishes on \(F_i\) for \(i < n\).

Let \(Z\) be a separable Lindenstrauss subspace of \(X\) which contains the sequence \(\{x_n\}_{n=0, 1, 2, \cdots}\) [7, Theorem 4.4 (a) and Theorem 6.1, (2)\(\Rightarrow\)(12)]. Let \(\varphi : X^*/Z^\perp \cong Z^*\) be the canonical map. For \(n \geq 1\), 
\[
P_n = \{z^* \in S(Z^*) : \langle z^*, x_n \rangle = 1\}
\]
is a weak*-
s closed face of \(S(Z^*)\) and \(\varphi(q_n) \in P_n\). There is a \(z_n^* \in \text{Ext} P_n \subseteq \text{Ext} S(Z^*)\) such that 
\[
|1 - \langle z_n^*, x_0 \rangle| \geq \langle \varphi(q_n), x_0 \rangle > 1 - 2^{-n}.
\]
Now if \(n < m\) we have \(\langle z_n^*, x_m \rangle = 0\) since, by the Hahn-Banach theorem, \(\varphi^{-1} \cap F_n \neq \emptyset\). Recalling that \(\langle z_n^*, x_n \rangle = 1\), we note that \(z_n^* \neq z_m^*\) for \(n \neq m\). Now if \(z^*\) is a weak*-
s limit point of \(\{z_n^*\}\) we have \(\langle z^*, x_0 \rangle = 1\).

**Corollary 2.7.** Let \(X\) be a simplex space. Then the following statements are equivalent:

1. There is no weak*-accumulation point \(q\) of \(\text{Ext} S(X^*)\) with \(\|q\| = 1\).
2. \(X\) is polyhedral.
3. No subspace of \(X\) is linearly isometric to \(c\).

**Proof.** We need only show that for each weak*-accumulation point \(q\) of \(\text{Ext} S(X^*)\) satisfying \(\|q\| = 1\), there is a \(v \in X\) satisfying 
\[
\langle q, v \rangle = 1 = \|v\|.
\]
There is no loss of generality in assuming \(q \in X^*\) since \(\text{Ext} S(X^*) \subseteq X^* \cup X^*^\perp\), and \(X^*\) and \(X^*^\perp\) are closed. Let \(F\) be the minimal weak*-closed face of \(K = S(X^*) \cap X^+\) containing \(q\). Clearly \(F\) consists of elements all of norm one. On the face \([0, 1]\) \(F\) define a weak*-continuous function
by $h(xf) = x$. As $K$ is a simplex, $h$ is well defined. Apply Edwards separation theorem to get an isometric extension $v \in X$ of the function $h$. Then $\langle q, v \rangle = 1 = \|v\|$

ACKNOWLEDGMENT. We would like to thank the referee for his many valuable comments. In particular, the extension of Theorem 1.2 to the nonseparable case is due to him.

REFERENCES