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COMPACT MULTILINEAR TRANSFORMATIONS

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ABSTRACT. It is well known that if a C^1 map between Banach spaces is compact, then its derivative is a compact operator. If the map is C^r, then what can be said about the compactness of its higher derivatives? This question leads us to a study of compact multilinear operators with the main result being that the higher derivatives of a compact map are such operators.

Let $E_1, \cdots, E_m$, F be Banach spaces, and $T:E_1\times\cdots\times E_m\to F$ be a continuous m-multilinear transformation. We call T *jointly compact* if it takes bounded sets into relatively compact sets, and we call T *separately compact* if for each j and any choice of points $e_i\in E_i$, $i\neq j$, the transformation $T(e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_m):E_j\to F$ is compact. The object of this paper is to expose several simple properties of such transformations. We give an example (§A) where separate compactness does not imply joint compactness, and we show that the first r derivatives of a compact C^r function are jointly compact (§B). We then look at linear spaces of jointly compact transformations (§C) and investigate their behavior with respect to tensor products (§D).

(A) *Let $T:l_2\times l_2\to l_2$ be defined as $T(a, b)=(a_1b_1, a_2b_2, \cdots)$ where $a=(a_1, a_2, \cdots)$ and $b=(b_1, b_2, \cdots)$ are in l_2. Then T is bilinear, continuous, and separately compact but not jointly compact.*

Continuity follows from the inequalities

$$\begin{aligned}\|T(a, b)\| &= \left(\sum a_i^2b_i^2\right)^{1/2} \leqq \left(\sum a_i^4\right)^{1/4}\left(\sum b_i^4\right)^{1/4}\\ &\leqq \left(\sum a_i^2\right)^{1/4}\left(\sum b_i^2\right)^{1/4} \leqq 1 \quad \text{for } \sum a_i^2, \sum b_i^2 \leqq 1.\end{aligned}$$

To show separate compactness fix $b\in l_2$ and take N so that $\sum_{N+1}^{\infty} b_i^2<\varepsilon$. If $\|a\|\leqq 1$, then

$$\begin{aligned}T(a, b) &= (a_1b_1, \cdots, a_Nb_N, 0, \cdots) + (0, \cdots, 0, a_{N+1}b_{N+1}, \cdots)\\ &\subset (\text{a closed } N\text{-dimensional ball of radius } \|b\|)\\ &\quad + (\text{a set of radius } < \varepsilon).\end{aligned}$$

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Therefore, the set $\{T(a, b): \|a\| \leqq 1\}$ has a finite ε-net for any $\varepsilon > 0$. On the other hand let $c^n = (0, \cdots, 0, 1, 0, \cdots)$ where 1 appears only in the nth slot. Then $T(c^n, c^n) = c^n$ has no convergent subsequences.

(B) *Let f be a C^r differentiable map of an open set U of a Banach space E into a Banach space F which takes bounded sets into relatively compact sets, then the derivatives $(D^nf)_x: E \times \cdots \times E \to F$ are jointly compact transformations for each $x \in U$ $(1 \leqq n \leqq r)$.*

The case $n=1$ appears in [**4**]. It is sufficient to show that the map $\phi: E \to F$ defined by $\phi(e) = (D^nf)_x(e, \cdots, e)$ is compact since by a polarization identity for symmetric transformations [**2**] we have

$$(D^nf)_x(e_1, \cdots, e_n) = \sum_{\varepsilon_1, \cdots, \varepsilon_n = 0}^{1} (-1)^{n-(\varepsilon_1 + \cdots + \varepsilon_n)} \phi(\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n).$$

Suppose ϕ is not compact. Then there is a sequence $\{e^N\}$ in the unit ball of E such that

$$\|(D^nf)_x(e^N, \cdots, e^N) - (D^nf)_x(e^M, \cdots, e^M)\| \geqq (n+2)\varepsilon \quad \text{for all } N \neq M.$$

Let

$$\begin{aligned} w(x, e) \equiv f(x + e) &- f(x) - (Df)_x(e) \\ &- \frac{1}{2!}(D^2f)_x(e, e) - \cdots - \frac{1}{n!}(D^nf)_x(e, \cdots, e); \end{aligned}$$

then by Taylor's formula [**1**] there is a δ such that $\|w(x, e)\| < \varepsilon \|e\|^n$ for all $\|e\| < \delta$. We now have

$$\begin{aligned} &\|f(x + \delta e^N) - f(x + \delta e^M)\| \\ &\quad = \Big\| \sum_{i=1}^{n} \frac{1}{i!}(D^if)_x(\delta e^N, \cdots, \delta e^N) \\ &\qquad\quad - \sum_{i=1}^{n} \frac{1}{i!}(D^if)_x(\delta e^M, \cdots, \delta e^M) + w(x, \delta e^N) - w(x, \delta e^M) \Big\| \\ &\quad \geqq \|(D^nf)_x(\delta e^N, \cdots, \delta e^N) - (D^nf)_x(\delta e^M, \cdots, \delta e^M)\| \\ &\qquad\quad - \sum_{i=1}^{n-1} \frac{1}{i!} \|(D^if)_x(\delta e^N, \cdots, \delta e^N) - (D^if)_x(\delta e^M, \cdots, \delta e^M)\| \\ &\qquad\quad - \|w(x, \delta e^N)\| - \|w(x, \delta e^M)\| \\ &\quad \geqq (n+2)\delta^n \varepsilon - \sum_{i=1}^{n-1} \frac{\delta^i}{i!} \|(D^if)_x(e^N, \cdots, e^N) \\ &\qquad\qquad - (D^if)_x(e^M, \cdots, e^M)\| - \delta^n \varepsilon - \delta^n \varepsilon. \end{aligned}$$

Utilizing the compactness of $(D^i f)_x$ $(1 \leqq i \leqq n-1)$ we pick a subsequence of $\{e^N\}$ denoted by $\{\bar{e}^N\}$ such that $\{(D^i f)_x(\bar{e}^N, \cdots, \bar{e}^N)\}$ are Cauchy sequences in N for each i. We can therefore find a P such that

$$\|(D^i f)_x(\bar{e}^N, \cdots, \bar{e}^N) - (D^i f)_x(\bar{e}^M, \cdots, \bar{e}^M)\| \leqq i!\, \delta^{n-i}\varepsilon$$

for $M, N \geqq P$ and $1 \leqq i \leqq n-1$. From the inequality above we obtain

$$\|f(x+\delta\bar{e}^N) - f(x+\delta\bar{e}^M)\| \geqq (n+2)\,\delta^n\varepsilon - (n-1)\,\delta^n\varepsilon - \delta^n\varepsilon - \delta^n\varepsilon = \delta^n\varepsilon$$

for M, $N \geqq P$. But this together with the fact that $\{x+\delta\bar{e}^N\}_{N=P}^{\infty}$ is a bounded sequence contradicts the compactness of f.

(C) Let $L(E_1, \cdots, E_m; F)$ be the Banach space of continuous m-multilinear transformations from $E_1 \times \cdots \times E_m$ to F, and let $CL(E_1, \cdots, E_m; F)$ be the closed subspace of jointly compact maps. It is easy to see that *the canonical Banach space isomorphism* $L(E_1, L(E_2, \cdots, L(E_m; F) \cdots)) \rightarrow L(E_1, \cdots, E_m; F)$ *induces the injection*

$$CL(E_1, CL(E_2, \cdots, CL(E_m; F) \cdots)) \rightarrow CL(E_1, \cdots, E_m; F),$$

however, this map is not necessarily surjective.

We let H be any real Hilbert space and show that the inclusion $CL(H, CL(H; R)) \subset CL(H, H; R)$ is proper. Define the bilinear map $T(a, b) = (a, b)$ where $(\ ,\ )$ is the inner product in l_2. Since $|(a, b)| \leqq \|a\|\,\|b\|$, we clearly have $T \in CL(H, H; R)$. But the map $a \rightarrow (a,\ )$ is not in $CL(H, CL(H; R))$ since it gives an isomorphism of H with its dual $H^* = CL(H; R)$ and, therefore, cannot be compact.

(D) *If E, F, G are Banach spaces and $E \hat{\otimes} F$ is the completion of $E \otimes F$ under the projective norm* [3], *then the canonical (algebraic) isomorphism $L(E, F; G) \rightarrow L(E \hat{\otimes} F; G)$ induces the isomorphism*

$$CL(E, F; G) \rightarrow CL(E \hat{\otimes} F; G).$$

Given any continuous bilinear T there is a continuous linear $\hat{T}$ such that the diagram

$$\begin{tikzcd} E \times F \arrow[rr, "\phi"] \arrow[dr, "T"'] & & E \hat{\otimes} F \arrow[dl, "\hat{T}"] \\ & G & \end{tikzcd}$$

commutes; ϕ is the obvious canonical map. The isomorphism above it given by the correspondence between T and $\hat{T}$. Suppose $\hat{T}$ is compact. Since $\|e \otimes f\| = \|e\|\,\|f\|$, we have that ϕ takes bounded sets to bounded sets which under $\hat{T}$ go into relatively compact sets. Therefore, $\hat{T}\cdot\phi = T$ is compact.

Conversely assume T is compact. Every element ω of the open unit ball of $E \times F$ has a representation $\omega = \sum_{n=0}^{\infty} \lambda_n (e_n \otimes f_n)$ where $\{e_n\}$ and $\{f_n\} \to 0$ in the open unit balls U and V of E and F respectively and $\sum_{n=0}^{\infty} |\lambda_n| < 1$ [3]. Therefore the closed ball of radius $\frac{1}{2}$ in $E \hat{\otimes} F$ is contained in the closure (cl) of the convex balanced hull (cbh) of $\phi(U \times V)$. Applying $\hat{T}$ we get

$$\begin{aligned}\hat{T}(\omega) \in \hat{T}(\mathrm{cl}(\mathrm{cbh}(\phi(U \times V)))) &\subset \mathrm{cl}(\hat{T}(\mathrm{cbh}(\phi(U \times V)))) \\ = \mathrm{cl}(\mathrm{cbh}(\hat{T}(\phi(U \times V)))) &= \mathrm{cl}(\mathrm{cbh}(T(U \times V))).\end{aligned}$$

The theorem of Mazur which says that the closed convex hull of a relatively compact subset of a Banach space is compact can also be stated for closed convex balanced hulls. Therefore we have that the image of the ball of radius $\frac{1}{2}$ under $\hat{T}$ is trapped in a compact set, and hence $\hat{T}$ is compact.

If $T_1 : E_1 \to F_1$ *and* $T_2 : E_2 \to F_2$ *are compact, then so is* $T_1 \hat{\otimes} T_2 : E_1 \hat{\otimes} E_2 \to F_1 \hat{\otimes} F_2$. This fact follows from the following commutative diagram

$$\begin{array}{ccc} E_1 \times E_2 & \xrightarrow{\phi_E} & E_1 \hat{\otimes} F_2 \\ T_1 \times T_2 \downarrow & & \downarrow T_1 \hat{\otimes} T_2 \\ F_1 \times F_2 & \xrightarrow[\phi_F]{} & F_1 \otimes F_2 \end{array}$$

the compactness of $T_1 \times T_2$, and the result above.

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