ANNIHILATOR IDEALS IN THE COHOMOLOGY OF BANACH ALGEBRAS

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Abstract. If $A$ is a $C^*$-algebra, if $X$ is a Banach $A$-module, and if $J$ is the annihilator of $X$ in $A$, then the cohomology space $\mathcal{H}^n(A, X^*)$ is isomorphic to $\mathcal{H}^n(A/J, X^*)$ for each positive integer $n$.

B. E. Johnson [5, Proposition 1.8] has shown that if $A$ is a Banach algebra with a bounded approximate identity and if $X$ is a Banach $A$-module, then $X_x = \{axb : x \in X; a, b \in A\}$ is a closed neo-unital submodule of $X$ and $\mathcal{H}^n(A, X^*)$ is isomorphic to $\mathcal{H}^n(A, X^*_x)$. In particular this result shows that in calculating cohomologies of dual Banach modules for $C^*$-algebras attention may be restricted to neo-unital modules. Since each closed (two-sided) ideal in a $C^*$-algebra has a bounded approximate identity [2, Propositions 1.8.2 and 1.7.2] our result shows that for $C^*$-algebras and dual Banach modules attention may be restricted to faithful modules. If $A$ is a Banach algebra, if $X$ is a Banach $A$-module, and if $J$ is a closed ideal in $A$ annihilating $X$, then there is a natural homomorphism $Q$, which is defined in Theorem 1, from $\mathcal{H}^n(A/J, X^*)$ into $\mathcal{H}^n(A, X^*)$. Under the additional assumption that $J$ has a bounded approximate identity, this homomorphism $Q$ is an isomorphism (Theorem 1). In Remark 4 we give an elementary example to show that an additional assumption on $J$ is necessary if the conclusion of Theorem 1 is to hold.

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If $A$ is a Banach algebra, if $X$ is a Banach $A$-module, and if $n$ is a positive integer, we let $\mathcal{L}^n(A, X^*)$ denote the Banach space of continuous $n$-linear mappings from $A$ into $X^*$, the dual of $X$ (our notation and definitions are from [5]). Recall that $\mathcal{L}^n(A, X^*)$ is the dual space of a Banach space $A \otimes A \otimes \cdots \otimes A \otimes X$ (see [5]). We also give $X^*$ the dual $A$-module structure from the Banach $A$-module $X$ by defining $af$ and $fa$ for $a$ in $A$ and $f$ in $X^*$ by

$$\langle af, x \rangle = \langle f, xa \rangle \quad \text{and} \quad \langle fa, x \rangle = \langle f, ax \rangle$$

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The mapping $\delta^n$ from $\mathcal{L}^{n-1}(A, X^*)$ to $\mathcal{L}^n(A, X^*)$ is defined by

$$
(\delta^n T)(a_1, \cdots, a_n) = a_1 T(a_2, \cdots, a_n) + \sum_{j=1}^{n-1} (-1)^j T(a_1, \cdots, a_j a_{j+1}, \cdots, a_n) + (-1)^n T(a_1, \cdots, a_{n-1}) a_n
$$

for $T$ in $\mathcal{L}^{n-1}(A, X^*)$, which we take to be $X^*$ when $n-1=0$, and for $a_1, \cdots, a_n$ in $A$. Then $\delta^{n+1} \delta^n = 0$, and we let

$$
\mathcal{H}^n(A, X^*) = \text{Ker} \delta^{n+1} / \text{Im} \delta^n.
$$

We use the same $\delta^n$ for all algebras and modules. We say that an ideal $J$ in a Banach algebra $A$ annihilates a Banach $A$-module $X$ if $ax = xa = 0$ for all $a$ in $J$ and $x$ in $X$. If the closed ideal $J$ annihilates $X$, we regard $X$ as a Banach $A/J$-module by defining $(a+J)x = ax$ and $(a+J)x = xa$ for all $a$ in $A$ and $x$ in $X$.

**Theorem 1.** Let $A$ be a Banach algebra, let $X$ be a Banach $A$-module, and let $J$ be a closed ideal in $A$ annihilating $X$. If $J$ has a bounded approximate identity, then $\mathcal{H}^n(A, X^*)$ is isomorphic to $\mathcal{H}^n(A, X^*)$ under the mapping $Q: T + \text{Im} \delta^n \to \theta T + \text{Im} \delta^n$

where $(\theta T)(a_1, \cdots, a_n) = T(a_1 + J, \cdots, a_n + J)$ for $T$ in $\mathcal{L}^n(A/J, X^*)$ and $a_1, \cdots, a_n$ in $A$.

We require a lemma before proving Theorem 1. Under the hypotheses of Theorem 1 we shall regard $\mathcal{L}^1(A/J, X^*)$ as a Banach $A$-module by defining $aT$ and $Ta$, for all $a$ in $A$ and $T$ in $\mathcal{L}^1(A/J, X^*)$, by

$$
(aT)(b + J) = aT(b + J) \quad \text{and} \quad (Ta)(b + J) = T(ab + J) - T(a + J)b
$$

for all $b$ in $A$. Compare the following lemma with [5, 1.a].

**Lemma 2.** Let $A$ be a Banach algebra, let $X$ be a Banach $A$-module, and let $J$ be a closed ideal in $A$ annihilating $X$. Let $n$ be an integer greater than 1. If $J$ has a bounded approximate identity, then $\mathcal{H}^{n-1}(A, \mathcal{L}^1(A/J, X^*))$ is isomorphic to $\mathcal{H}^n(A, X^*)$ under the mapping $\chi: T + \text{Im} \delta^{n-1} \to \psi_n T + \text{Im} \delta^n$

where

$$
(\psi_n T)(a_1, \cdots, a_n) = T(a_1, \cdots, a_{n-1})(a_n + J)
$$

for all $T$ in $\mathcal{L}^{n-1}(A, \mathcal{L}^1(A/J, X^*))$ and all $a_1, \cdots, a_n$ in $A$. 
Proof. A routine calculation using equations (1), (2), and (3) shows that

\[ \psi_{n+1}\delta^n = \delta^{n+1}\psi_n \]

for \( n \) a positive integer. Applying equation (4) with \( n \) replaced by \( n-1 \) we observe that the mapping \( \chi \) is well defined. Using (4) as it stands we observe that \( \chi \) maps \( \mathcal{A}^{n-1}(A, \mathcal{L}^2(A/J, X^*)) \) into \( \mathcal{A}^n(A, X^*) \). We now use the bounded approximate identity in \( J \) to show that \( \chi \) is an isomorphism.

We shall show that there is an \( R \) in \( \mathcal{L}^{n-1}(A, X^*) \) with

\[ (T - \delta^nR)(a_1, \ldots, a_n) = 0 \]

if \( a_n \) is in \( J \). If we have found such an \( R \), then equation (5) implies that \( T - \delta^nR \) is in the image of \( \psi_n \), and so there is a \( P \) in \( \mathcal{L}^{n-1}(A, \mathcal{L}^2(A/J, X^*)) \) with \( \psi_nP = T - \delta^nR \). From equation (4) we obtain

\[ \psi_{n+1}\delta^nP = \delta^{n+1}\psi_nP = \delta^{n+1}(T - \delta^nR) = 0, \]

and thus \( \delta^nP = 0 \) because \( \psi_{n+1} \) is a monomorphism. Therefore \( \chi \) is an epimorphism.

Let \( T \) be in \( \mathcal{L}^n(A, X^*) \) with \( \delta^{n+1}T = 0 \), and let \( \{e_a\} \) be a bounded approximate identity in \( J \). Now \( \mathcal{L}^n(A, X^*) \) may be regarded as the dual space of \( A \otimes \cdots \otimes A \otimes X \), where there are \( n \)-copies of \( A \), and under this identification the weak-* topology on \( \mathcal{L}^n(A, X^*) \) is generated by the seminorms \( S \rightarrow (S(a_1, \ldots, a_n), x) \) where \( \langle \cdot, \cdot \rangle \) denotes the pairing of \( X \) and \( X^* \), and \( a_1, \ldots, a_n \) are in \( A \) and \( x \) is in \( X \) (see [5, §1]). Now \( \{T(\cdot, \ldots, e_a)\} \) is a bounded net in \( \mathcal{L}^{n-1}(A, X^*) \) and hence has a subnet convergent in the weak-* topology of \( \mathcal{L}^{n-1}(A, X^*) \). For convenience, we take \( \{T(\cdot, \ldots, e_a)\} \) itself to be convergent in the weak-* topology to an element \( (-1)^{n+1}R \) in \( \mathcal{L}^{n-1}(A, X^*) \). All limits in this proof are over the directed set corresponding to the net \( \{e_a\} \) and are in the weak-* topology in \( X^* \).

Using \( \delta^{n+1}T(a_1, \ldots, a_n, e_a) = 0 \), equation (1), and the definition of \( R \), we obtain

\[ \delta^nR(a_1, \ldots, a_n) = (-1)^{n+1}\lim \left\{ a_1T(a_2, \ldots, a_n, e_a) + \sum_{j=1}^{n-1} (-1)^jT(a_1, \ldots, a_ja_{j+1}, \ldots, a_n, e_a) \right\} \]

\[ + (-1)^nT(a_3, \ldots, a_{n-1}, e_a)a_n \]

\[ = (-1)^n \lim \{(-1)^nT(a_1, \ldots, a_{n-1}, e_a) + (-1)^{n+1}T(a_1, \ldots, a_n, e_a) \}
\]

\[ + (-1)^{n+1}T(a_1, \ldots, a_{n-1}, e_a)a_n \]

\[ = T(a_1, \ldots, a_n) \]

provided \( a_n \) is in \( J \). This proves equation (5).
We shall now prove that \( \chi \) is one-to-one. We let \( T \) be in
\[
\mathcal{L}^{n-1}(A, \mathcal{L}^1(A/J, X^*))
\]
with \( \psi_n T = \delta^n S \) where \( S \) is some element of \( \mathcal{L}^{n-1}(A, X^*) \). We shall obtain an \( R \) in \( \mathcal{L}^{n-2}(A, X^*) \) such that
\[
(S - \delta^{n-1} R)(a_1, \ldots, a_{n-1}) = 0
\]
if \( a_{n-1} \) is in \( J \). Having found such an \( R \) there is a \( P \) in \( \mathcal{L}^{n-2}(A, \mathcal{L}^1(A/J, X^*)) \) such that \( \psi_{n-1} P = S - \delta^{n-1} R \). From this and equation (4) we obtain
\[
\psi_n T = \delta^n S = \delta^n \psi_{n-1} P + \delta^n \delta^{n-1} R = \psi_n \delta^{n-1} P.
\]
Because \( \psi_n \) is a monomorphism, \( T \) is equal to \( \delta^{n-1} P \).

We let \( T \) be in \( \mathcal{L}^{n-1}(A, \mathcal{L}^1(A/J, X^*)) \) with \( \psi_n T = \delta^n S \) where \( S \) is some element of \( \mathcal{L}^{n-1}(A, X^*) \). As in the above proof that \( \chi \) is an epimorphism, there is a bounded approximate identity \( \{e_n\} \) in \( J \) and an \( R \) in \( \mathcal{L}^{n-2}(A, X^*) \) such that
\[
R(a_1, \ldots, a_{n-2}) = (-1)^n \lim S(a_1, \ldots, a_{n-2}, e_n)
\]
for all \( a_1, \ldots, a_{n-2} \) in \( A \). If \( a_{n-1} \) is in \( J \) and if \( a_1, \ldots, a_{n-2} \) are in \( A \), then
\[
\delta^{n-1} R(a_1, \ldots, a_{n-1})
= (-1)^n \lim \left\{ a_1 S(a_2, \ldots, a_{n-1}, e_n) + \sum_{j=1}^{n-2} (-1)^j S(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n-1}, e_n) 
+ (-1)^{n-1} S(a_1, \ldots, a_{n-2}, e_n) a_{n-1} \right\}
= (-1)^n \lim \left\{ \delta^n S(a_1, \ldots, a_{n-1}, e_n) + (-1)^n S(a_1, \ldots, a_{n-2}, a_{n-1} e_n) \right\}
\]
because \( a_{n-1} \) and \( e_n \) are in \( J \), which annihilates \( X^* \). Since \( \delta^n S = \psi_n T \), it follows that \( \delta^n S(a_1, \ldots, a_{n-1}, e_n) = 0 \) because \( e_n \) is in \( J \). This completes the proof of the lemma.

**Proof of Theorem 1.** The definitions of \( \theta \) and \( \delta^n \) imply that \( \delta^n \theta = \theta \delta^n \). Thus the mapping \( \theta \), defined in the statement of the theorem, is a well defined homomorphism from \( \mathcal{H}^n(A/J, X^*) \) into \( \mathcal{H}^n(A, X^*) \). We shall prove that \( \theta \) is an isomorphism by induction on \( n \) over all Banach \( A \)-modules that are annihilated by \( J \).

Now we consider \( n=1 \). If \( f \) is in \( X^* \), then \( \delta^1(f)(a) = af - fa = (a+J)f - f(a+J) = \delta^1(f)(a+J) = (\theta \delta^1(f))(a) \) by definition of \( \delta^1 \), so that \( \theta \) Im \( \delta^1 = \text{Im} \delta^1 \). If \( D \) is in \( \text{Ker} \delta^1 \), which is contained in \( \mathcal{L}^1(A, X^*) \), then \( D(ab) = D(a)b + aD(b) \) for all \( a, b \) in \( A \) by definition of \( \delta^2 \). If \( c \) is in \( J \), then by Cohen's Factorization Theorem [1] we have \( c = ab \) for some \( a \) and \( b \) in \( J \). Thus \( D(c) = aD(b) + D(a)b = 0 \) because \( J \) annihilates \( X^* \). We may now define an operator \( \theta \) in \( \mathcal{L}^1(A/J, X^*) \) by \( \theta(a+J) = D(a) \) for all \( a \) in \( A \). Then \( \theta \) is \( D \), and \( \delta^2 T = 0 \). This shows that \( \theta \) is an isomorphism for \( n=1 \).
Suppose the result has been proved for \( n \). We firstly observe that 
\( L^0(A/J, X^*) \) is, as a Banach \( A \)-module, the dual of the Banach \( A \)-module 
\( Y = (A/J) \hat{\otimes} X \), the projective tensor product of Banach spaces [5, §1],
where we define the module operations on generating elements \((a+J) \hat{\otimes} x\) of the tensor product by

\[
\begin{align*}
    b((a + J) \hat{\otimes} x) &= (ba + J) \hat{\otimes} x \quad \text{and} \\
    ((a + J) \hat{\otimes} x)b &= (a + J) \hat{\otimes} xb
\end{align*}
\]

and lift the definitions to \( Y \) by linearity and continuity. Because \( J \) annihilates \( X \), equations (7) imply that \( J \) annihilates the \( A \)-module \( Y \). Now by Lemma 2, our inductive hypothesis on \( n \), and the reduction of dimension lemma for cohomology [5, 1(a)], the following isomorphisms hold:

\[
\mathcal{H}^{n+1}(A, X^*) \cong \mathcal{H}^{n}(A, L^0(A/J, X^*)) \cong \mathcal{H}^{n}(A/J, L^0(A/J, X^*)) \cong \mathcal{H}^{n+1}(A/J, X^*). 
\]

Each of these isomorphisms is the natural one arising from the quotient \( A/J \). Thus \( Q \) is an isomorphism for \( n+1 \). This completes the proof.

Our corollary generalizes [4, Theorems 4.1 and 4.2] from \( n=1 \) and 2 to any positive integer \( n \).

**Corollary 3.** Let \( A \) be a Banach algebra in which each closed cofinite ideal has a bounded approximate identity. If \( X \) is a finite dimensional Banach \( A \)-module and \( n \) is a positive integer, then \( J^n(A, X) = \{0\} \).

**Proof.** The annihilator \( J \) of \( X \) is a closed cofinite ideal in \( A \), and \( X \) is the dual of the Banach \( A \)-module \( X^* \). By Theorem 1, we have \( \mathcal{H}^n(A, X) \) isomorphic to \( \mathcal{H}^n(A/J, X) \). An ideal in \( A/J \) is of the form \( I/J \), where \( I \) is a closed cofinite ideal in \( A \) containing \( J \). Since \( I \) has a bounded approximate identity, \( I/2 \) is equal to \( I \) by Cohen's Factorization Theorem [1]. This shows that \( A/J \) is a finite dimensional semisimple algebra. As every \( n \)-linear operator from \( A/J \) into \( X \) is continuous, \( \mathcal{H}^n(A/J, X) \) coincides with Hochschild's cohomology groups [3] for the \( A/J \)-module \( X \). Hochschild's \( n \)th-cohomology group for the \( A/J \)-module \( X \) is null [3, Theorem 4.1], and so \( \mathcal{H}^n(A, X) = \{0\} \).

**Remark 4.** We now outline an example which shows that some assumption on \( J \) like that of a bounded approximate identity is necessary if the conclusion of Theorem 1 is to hold. Let \( X \) be a (finite dimensional) Banach space, and let \( X \) have the zero product (\( xy=0 \) for all \( x, y \) in \( X \)). Let \( A \) be the Banach algebra obtained by adjoining an identity to \( X \), and let the ideal \( J \) be \( X \). We regard \( X \) as an \( A \)-module with the natural module operations. Then \( A/J \) is equal to \( C^1 \), and so \( \mathcal{H}^1(A/J, X^*) \) is zero as may be proved in a number of ways (for example [3, Theorem 4.1]). However
for the algebra $A$ we obtain $\text{Im} \, \delta^1$ is $\{0\}$, and $\text{Ker} \, \delta^2$ is $\mathcal{L}^1(J, X^*)$, so that $\mathcal{A}^1(A, X^*) = \mathcal{L}^1(J, X^*)$ and the conclusion of Theorem 1 does not hold.

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