PL INVOLUTIONS OF SOME 3-MANIFOLDS

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Abstract. Let $h_1$ and $h_2$ be PL involutions of connected, oriented, closed, irreducible 3-manifolds $M_1$ and $M_2$, respectively. Let $a_i, i = 1, 2$, be a fixed point of $h_i$ such that near $a_i$ the fixed point sets of $h_i$ are of the same dimension. Then we obtain a PL involution $h_1 \# h_2$ on $M_1 \# M_2$ induced by $h_i$ by taking the connected sum of $M_1$ and $M_2$ along neighborhoods of $a_i$. In this paper, we study the possibility for a PL involution $h$ on $M_1 \# M_2$ having a 2-dimensional fixed point set $F_0$ to be of the form $h_1 \# h_2$, where $M_i$ are lens spaces. It is shown that: (1) if $F_0$ is orientable, then $M_1 = -M_2$ and $h$ is the obvious involution, (2) if the fixed point set $F$ contains a projective plane, then $M_1 = M_2 = a$ projective 3-space, and in this case, $F$ is the disjoint union of two projective planes and $h$ is unique up to PL equivalences, (3) if $F$ contains a Klein bottle $K$, then $F$ is the disjoint union of a Klein bottle and two points.

1. Introduction. Let $M$ be a closed, oriented 3-manifold which is the connected sum $M_1 \# M_2$ of two irreducible 3-manifolds $M_1$ and $M_2$, and let $h$ be a PL involution of $M$ with a fixed point set $F$ containing a non-orientable surface $F_0$. Since $F_0$ is one-sided, it would seem that $h$ cannot interchange $M_i$-part and $M_j$-part and $h$ must be obtained from involutions $h_i$ of $M_i, i = 1, 2$, by attaching two involutions along an invariant neighborhood of fixed points $a_i$ of $h_i$, where near $a_i$ the fixed point sets are of the same dimension.

Kwun [4] proved that no lens spaces except the real projective space $P_3$ admits orientation reversing PL involutions, and in case of $P_3$ there exists a unique PL involution up to PL equivalences and $F$ is the disjoint union of a projective plane and a point. Hence it leads us to consider the possibility of the above question when $M_i$ are isomorphic to lens spaces.

2. PL involutions of lens spaces. Let $h_1$ and $h_2$ be PL involutions of $M_1$ and $M_2$, respectively, where $M_i, i = 1, 2$, is isomorphic to a lens space (not necessarily having the natural orientation). The connected sum $M = M_1 \# M_2$...
$M_1 \# M_2$ is obtained by removing the interior of a nice 3-cell from each, and then matching the resulting boundaries using an orientation reversing homeomorphism.

If $Z_2$ acts on $M$, $H^*(M/Z_2; \mathbb{Q}) \approx H^*(M; \mathbb{Q})^Z$, and since the involution reverses orientation, we obtain $H^3(M; \mathbb{Q})^Z = 0$, and hence $\chi(M/Z_2) = 1$. Therefore $2\chi(M/Z_2) = \chi(F) + \chi(M)$ implies $\chi(F) = 2$. Let $b_i(F)$ denote the $i$th mod 2 betti number of $F$. Then $\sum b_i(F) \leq \sum b_i(M) \leq 1 + 2 + 2 + 1 = 6$ (see [2, p. 42]). This, together with $\chi(F) = 2$ and $\dim F$ is even, implies that the possible fixed point sets are $S^2$, the disjoint union of two projective planes, the disjoint union of a projective plane and a point, the disjoint union of a Klein bottle and two points, the disjoint union of a Klein bottle and $S^2$, the disjoint union of $S^1 \times S^1$ and $S^2$, and the disjoint union of $S^1 \times S^1$ and two points.

Now we first consider the case that $F$ contains an orientable surface.

**Theorem 1.** Let $h$ be a PL involution of $M = M_1 \# M_2$. If the fixed point set $F$ contains an orientable surface, then $F$ is a 2-sphere and $M_2 = -M_1$, $h$ being the obvious involution.

**Proof.** Let $S$ be an orientable surface in $F$. By the Alexander duality theorem [9], over the rationals $\mathbb{Q}$, $F$ separates $M$ into two parts $U$ and $V$. Hence $M = 2\hat{U}$, and we have $U \hookrightarrow M \hookrightarrow U$, such that $r$ is the identity, where $i$ is the inclusion and $r$ is a retraction. Hence we obtain the exact sequence

$$H_i(U; \mathbb{Q}) \xrightarrow{i_*} H_i(M; \mathbb{Q}) \xrightarrow{r_*} H_i(\hat{U}; \mathbb{Q})$$

where $r_*i_*$ is the identity. Hence $H_i(U; \mathbb{Q}) = 0$ for $i = 1, 2$, and therefore $F$ must be a 2-sphere.

By Milnor [8], we may say that $U$ is isomorphic to $M_1$-part and $V$ is isomorphic to $M_2$-part, and since $U$ and $V$ must be interchanged by $h$, $M_2 = -M_1$ and $h$ is the obvious involution. This proves the theorem.

Hence we have eliminated the case that $F$ is the disjoint union of $S^1 \times S^1$ and $S^2$, the disjoint union of $S^1 \times S^1$ and two points, or the disjoint union of a Klein bottle and $S^2$.

Since the case where a 2-dimensional component of $F$ is orientable has been taken care of, we have to consider only the case where each 2-dimensional component of $F$ is nonorientable.

**Theorem 2.** Let $M$ be $M_1 \# M_2$, where $M_i$ is isomorphic to a lens space, and $h$ a PL involution on $M$. If a real projective plane $P_2$ is contained in the fixed point set $F$ of $h$, then $M = P_3 \# P_3$.

**Proof.** Suppose that $h$ fixes a real projective plane $P$ and assume that $M$ has been triangulated so that $h$ is simplicial and the simplicial neighborhood $U$ of $P$ is an invariant regular neighborhood of $P$. Moreover,
assume that $h|_{U-P}$ is fixed point free. Let $p:N_1 \to M$ be the double covering obtained from $M$ by cutting along $P$. Since $U$ is orientable, but $P$ is not, $(U, P)$ is homeomorphic to $(N, P)$, where $N$ is the mapping cylinder of a double covering $S^3 \to P$. Let $U'$ and $(M-U)'$ be the connected manifolds obtained from $U$ and $(M-U)$ by attaching a 3-cell to each. Then by Milnor [8], $U'$ is isomorphic to $S^3$, $M_1$, or $M_2$. But since $\pi_1(U')$ is $Z_2$, $U'$ cannot be isomorphic to $S^3$, and we may say $U'$ is isomorphic to $M_1$. Now $h|_{U'}$ can be extended to an orientation reversing PL involution $h'$ of $U'$ defined by the cone over $h|_{\partial(U')}$. Then by Kwun [4], $U' \approx M_1 \approx P_3$. Similarly $M_2 \approx P_3$. This completes the proof.

Theorem 3. Let $M=M_1 \# M_2$ and $h$ a PL involution of $M$. If a projective plane is contained in the fixed point set $F$ of $h$, then $F$ is the disjoint union of two projective planes, and $h$ being of the form $h_1 \# h_2$ is unique.

Proof. Suppose that a projective plane $P$ is contained in $F$ and assume that $h$ is simplicial, the simplicial neighborhood $U$ of $P$ is invariant regular, and $h|_{U-P}$ is fixed point free. We have seen in Theorem 2 that $h|_{M-U}$ can be extended to an orientation reversing PL involution $h'$ of $P_3$. Let $F'$ be the fixed point set of $h'$. Then by Kwun [3], $F'$ is the disjoint union of a projective plane and a point $p$. By the way we extended $h$ to $h'$, $p$ is the cone vertex. Therefore $h|_{M-U}$ has a projective plane as the fixed point set $F'$. Hence $F$ is the disjoint union of two projective planes.

Let $P$ and $P'$ be the two projective planes whose union is $F$. Take $U$ and $U'$ to be disjoint invariant regular neighborhoods of $P$ and $P'$, respectively. Then by Milnor [8], $M-(U \cup U')$ is isomorphic to $S^3 \times [0,1]$, and by Livesay [7], there is a unique involution on $S^3 \times [0,1]$ up to PL equivalences. Hence it suffices to analyse $h|_{U}$ and $h|_{U'}$. But by Kwun [3] these are unique, and $h=h_1 \# h_2$, where $h_1$ and $h_2$ are the extensions of $h|_{U}$ and $h|_{U'}$ to $P_3$. This proves the theorem.

From now on we consider the case that $F$ contains a Klein bottle $K$. The only possible $F$ is the disjoint union of $K$ and two points. The union of $K$ and two points can be the fixed point of an involution $h$. For, let the disjoint union of a projective plane $A_i$ and a point $p_i$, $i=1, 2$, be the fixed point set of $h_i$ of $P_3$. Taking the connected sum $P_3 \# P_3$ along an invariant neighborhood of $a_i, a_i \in A_i$, we obtain a PL involution $h_1 \# h_2$ whose fixed point set is the disjoint union of a Klein bottle and two points. Hence we have

Theorem 4. Let $M=M_1 \# M_2$ and $h$ a PL involution of $M$ with a Klein bottle in the fixed point set $F$. Then $F$ is the disjoint union of a Klein bottle and two points.
The uniqueness question for $h$ in case $F$ is the disjoint union of a Klein bottle and two points is not settled, but the following theorem gives some idea how the Klein bottle is located in $P_3 \# P_3$.

**Theorem 5.** Let $M$ be the connected sum $P_3 \# P_3$ and let $K$ be the Klein bottle in the fixed point set $F$ of an involution $h$ of $M$. Then $\pi_1(M - U)$ is the integers $\mathbb{Z}$ and $M - U$ is homeomorphic to $D^2 \times S^1$.

For the proof of Theorem 5, we first prove the following lemma.

**Lemma 6.** Let $h$ be a PL involution of $M = P_3 \# P_3$. Then there exists a PL involution $h' : S^1 \times S^2 \to S^1 \times S^2$ such that $p'h' = hp'$, where $p' : S^1 \times S^2 \to P_3 \# P_3$ is a 4-to-1 covering projection.

**Proof.** Consider the usual 2-to-1 covering map $p : S^1 \times S^2 \to M$. Let $H = h_\mu\pi_1(S^1 \times S^2)$ and $G = \pi_1(M)$. Suppose $h_\mu H \neq H$. Since $[G:H] = [G:h_\mu H] = 2$, there is no inclusion relation between $H$ and $h_\mu H$. Let $L = H \cap h_\mu H$. Then $L$ is a normal subgroup of $G$, and $H/L = H \cdot h_\mu H/h_\mu H = G/h_\mu H = \mathbb{Z}_2$ implies $[H:L] = 2$. Hence $[G:L] = 4$. Furthermore $h_\mu L = L$. Hence by the lifting theorem there is a PL involution $h'$ on $S^1 \times S^2$ such that $p'h' = hp'$, where $p'$ is 4-to-1. If $H = h_\mu H$, then the construction is similar and easier. This proves the lemma.

**Proof of Theorem 5.** Consider the double covering $p : N \to M$ obtained from $M$ cutting along $K$. Then $p^{-1}(K)$ is homeomorphic to $S^1 \times S^1$. Since each component of $M - p^{-1}(K)$ maps homeomorphically onto $M - K$, $p^{-1}(U)$ is a collar of $p^{-1}(K)$. So $\text{Bd}(U)$ is homeomorphic to $S^1 \times S^1$. By the Mayer-Vietoris sequence

$$H_2(M) \to H_1(S^1 \times S^1) \to H_1(U) \oplus H_1(M - U) \to H_1(M) \to 0$$

we obtain that $H_1(M - U)$ is a group of rank 1.

Now we show that $\pi_1(M - U)$ is the integers $\mathbb{Z}$. Let $p : S^1 \times S^2 \to P_3 \# P_3$ be a 4-to-1 covering such that $h_\mu p_\mu \pi_1(S^1 \times S^2) = p_\mu \pi_1(S^1 \times S^2)$. Then $h' : S^1 \times S^2 \to S^1 \times S^2$ exists such that the following diagram commutes:

$$S^1 \times S^2, \bar{x}_0 \quad \xrightarrow{h'} \quad S^1 \times S^2, \bar{x}_0$$

$$\downarrow p \quad \quad \quad \downarrow p$$

$$P_3 \# P_3, x_0 \quad \xrightarrow{h} \quad P_3 \# P_3, x_0 \quad (x_0 \in K).$$

Consider the component $A$ of $\bar{x}_0$ in $p^{-1}(K)$. Then $A$ is either $S^2 \times S^1$ or a Klein bottle $K$. 

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Case (1). If $A = S^1 \times S^1$, $h'$ fixes $S^1 \times S^1$ and $S^1 \times S^2 - A = R^2 \times S^1 \cup R^2 \times S^1$ [5]. The covering $A \to K$ is at least 2-to-1. If it is actually 2-to-1, then $p^{-1}(K) = A \cup B$, where $B$ is another $S^1 \times S^1$. Then $B$ is contained in one of the two $R^2 \times S^1$. But $h'$ interchanges two $R^2 \times S^1$, while $h'(p^{-1}(K)) = p^{-1}(K)$ and therefore $h'(B) = B$ which is impossible. Hence $p^{-1}(K) = A$. Hence each of the two $R^2 \times S^1$ double cover $P_3 \# P_3 - K$. Hence we have an exact sequence

$$0 \to Z \xrightarrow{f} \pi_1(M - U) \xrightarrow{g} Z_2 \to 0.$$  

Hence if we choose $x \in \pi_1(M - U)$ such that $g(x) \neq 0$ and a generator $y$ in $Z$, then $\pi_1(M - U)$ is generated by $x$ and $y$. Suppose $yx^{-1} = y^{-1}$. By abelianizing we have $y^2 = 1$. And since $x$ is of finite order too, $H_1(M - U)$ is finite which contradicts the fact that $H_1(M - U)$ is of rank 1. Hence $\pi_1(M - U)$ is abelian, and hence $\pi_1(M - U) = Z + \text{torsion}$ part. But since we have a universal covering $R^2 \times R^1 \to M - U$ and no nontrivial finite group can act freely on a finite-dimensional, contractible space, $\pi_1(M - U) = Z$.

Case (2). If $A = K$, then $h'$ fixes only $A$ and $S^1 \times S^2 - A = R^2 \times S^1$. If $p^{-1}(K)$ had any other component $B$, then $B \cap K$ and $B \cap R^2 \times S^1 \subseteq R^2$. But $K$ cannot be embedded in $R^2$. Hence $p^{-1}(K) = A$. Then $R^2 - A \approx R^2 \times S^1$ covers $P_3 \# P_3$ 4-to-1. Hence

$$0 \to Z \to \pi_1(M - U) \xrightarrow{\zeta} Z_2 + Z_2 \to 0$$

is exact, or

$$0 \to Z \to \pi_1(M - U) \xrightarrow{\zeta} Z_4 \to 0$$

is exact. Let $N$ be a subgroup of $Z_2 + Z_2$ (or $Z_4$) such that $N \approx Z_2$. Let $p^* : X \to M - U$ be the double covering corresponding to $\zeta^{-1}(N)$. Then there exists a double covering $p^* : R^2 \times S^1 \to X$. Then the argument in Case (1) shows that $\pi_1 X \approx Z$. Now consider the double covering $p^* : X \to M - U$. Repeating the same argument in Case (1), we get $\pi_1(M - U) = Z$.

Since $M - U$ is an irreducible, orientable, compact 3-manifold with $\pi_1(M - U) = Z$ and $\text{Bd}(M - U)$ is homeomorphic to $S^1 \times S^1$, $M - U$ is homeomorphic to $D^2 \times S^1$. This proves the theorem.

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References


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