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ON A CONSTRUCTION OF BREDON

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ABSTRACT. Using a homotopy-theoretical description of a geometric pairing due to Bredon, we show how to rederive Bredon's results on the pairing. Furthermore, we are able to, in some sense, complete these results by combining the homotopy-theoretical approach with Sullivan's determination of the 2-primary Postnikov decomposition of the space G/PL.

1. **Introduction.** In [1], Bredon introduced a geometric pairing

$$\Gamma_n \times \Pi_{n+k}(S^n) \xrightarrow{\rho_{n,k}} \Gamma_{n+k},$$

Γ_i being the group of differential structures on the i-sphere[2] and used this pairing to construct certain semifree actions of groups on spheres. There is also the pairing

$$\Pi_n(PL/O) \times \Pi_{n+k}(S^n) \xrightarrow{\gamma_{n,k}} \Pi_{n+k}(PL/O)$$

defined by composition, PL/O being the fibre of the natural map $BO \to BPL$ of classifying spaces. Since, by smoothing theory, $\Pi_n(PL/O)$ and Γ_n are isomorphic groups, it is natural to inquire about the relation between these two pairings. In fact, as observed by Bredon [**1**] and the present author [**3**], these two pairings coincide.

While Bredon's geometric definition is rather natural and simple, the main results and proofs are more perspicuous using the homotopy-theoretical definition. Thus, in §2 we show how to retrieve (and slightly generalize) the theorems of [**1**, §§1, 2] in this context, in particular the following important result, which we state as

THEOREM A [**1**, THEOREM 2.1]. *Let* $\alpha \in \Pi_{n+k}(S^n)$, $\sigma \in \Gamma_n$ *and let* $p_l(\sigma) \subset \Pi_{n+l}(S^l)$ *denote the set of all elements represented (via the Pontryagin-Thom construction) by framed embeddings* $\Sigma^n \times D^l \subset S^{n+l}$, Σ^n

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[2] Actually, Bredon defines his map from $\Theta_n \times \Pi_{n+k}(S^n)$ to Θ_{n+k}, but it is easily seen that this map restricts to a map from $\Gamma_n \times \Pi_{n+k}(S^n)$ to Γ_{n+k}.

a homotopy sphere representing σ. *Then*

$$p_l(\sigma) \circ \Sigma^l \alpha \subset p_l(\rho_{n,k}(\sigma, \alpha)),$$

where $\Sigma^l \alpha$ *is the lth iterated suspension of* α.

However, the main contribution of this note is the following Theorem B (essentially conjectured by Bredon [**1**, Corollary 2.3 and succeeding remarks]) which, taken together with Theorem A, gives a rather good hold on Bredon's pairing.

THEOREM B. *If* bP_{n+1} *is the subset of* Γ_n *consisting of those homotopy spheres which bound* Π-*manifolds, then the pairing* $\rho_{n,k}$ *restricts to a pairing* $\bar{\rho}_{n,k}: bP_{n+1} \times \Pi_{n+k}(S^n) \to bP_{n+k+1}$. *Moreover*, $\bar{\rho}_{n,k}$ *is trivial for* $k>0$.

The proof of Theorem B, which relies on results of Sullivan [**4**], will also be carried out in §2.

As a final introductory comment, we remark that while the main interest in the composition pairing $\Pi_n(PL/O) \times \Pi_{n+k}(S^n) \to \Pi_{n+k}(PL/O)$ lies in its geometric interpretation, the pairing has also proved to be of some use in studying the k-invariants of PL/O ([**5**], [**3**]).

2. **The main properties of** $\rho_{n,k}$. In this section we deduce the main properties of the pairing $\rho_{n,k}$, working in the homotopy-theoretical context. We begin with a proposition which summarizes and slightly generalizes the results of [**1**, §1].

PROPOSITION. *The pairing* $\rho_{n,k}: \Gamma_n \times \Pi_{n+k}(S^n) \to \Gamma_{n+k}$ *is bilinear and associative in the sense that the diagram*

$$\begin{tikzcd}
\Gamma_n \times \Pi_{n+k}(S^n) \times \Pi_{n+k+l}(S^{n+k}) \arrow[r, "\mathrm{id} \times \mathrm{comp}"] \arrow[d, "\rho_{n,k} \times \mathrm{id}"] & \Gamma_n \times \Pi_{n+k+l}(S^n) \arrow[d, "\rho_{n,k+l}"] \\
\Gamma_{n+k} \times \Pi_{n+k+l}(S^{n+k}) \arrow[r, "\rho_{n+k,l}"] & \Gamma_{n+k+l}
\end{tikzcd}$$

is commutative.

PROOF. Using the identification of $\rho_{n,k}$ with $\gamma_{n,k}$, all these statements, except perhaps the linearity in the first variable, are trivial. As for linearity in the first variable, this follows from the fact that PL/O is an H-space.[3]

We remark that linearity in the first variable was proved in [**1**] only under the assumption that $\Pi_{n+k}(S^n)$ is stable, i.e. $k<n-1$. Our proof, exploiting the H-structure on PL/O, shows this restriction on k to be unnecessary.

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[3] The bilinearity of $\gamma_{n,k}$ has been independently observed by Schultz (*Smooth structures on* $S^p \times S^q$, Ann. of Math. (2) **90** (1969), 187–198) in a closely related context.

We come now to the two key results, Theorems A and B.

PROOF OF THEOREM A. Let $\Gamma^f_{n+l,n}$ denote the set of framed embeddings of homotopy n-spheres in S^{n+l}, G_l^+ the space of maps $S^{l-1}\to S^{l-1}$ of degree 1. In addition to the obvious (forgetful) map $\varphi_i\cdot\Gamma^f_{i+l,i}\to\Gamma_i$, there is a map $\omega_i:\Gamma^f_{i+l,i}\to\Pi_i(G_l^+)$ described in [2].[4] It is not difficult to see that the set $p_l(\sigma)$, as originally defined by Kervaire-Milnor, can equivalently be described as $J_n(\omega_n(\varphi_n^{-1}(\sigma)))$, where $J_n:\Pi_n(G_l^+)\to\Pi_{n+l}(S^l)$ is obtained, as usual, by the Hopf construction. Thus, if $\beta\in p_l(\sigma)$, there exists $\eta\in\omega_n(\varphi_n^{-1}(\sigma))$ such that $\beta=J_n(\eta)$. But by a known formula (cf. [1, p. 442]) we have $J_n(\eta)\circ\Sigma^l\alpha=J_{n+k}(\eta\circ\alpha)$. Moreover, it is not difficult to see, for example by using the homotopy-theoretical interpretation of $\Gamma^f_{i+l,i}$, φ_i, ω_i (see footnote 4) that $\eta\circ\alpha\in\omega_{n+k}(\varphi^{-1}_{n+k}(\rho_{n,k}(\sigma,\alpha))$ and the theorem follows.

PROOF OF THEOREM B. If G_k is the set of maps $S^{k-1}\to S^{k-1}$ of degree ± 1 and $G=\lim_{k\to\infty}G_k$, there are natural maps $O\subset PL\subset G$ and a fibration $PL/O\to G/O\to G/PL$. For $\alpha\in\Pi_{n+k}(S^n)$, we consider the diagram.

$$\begin{array}{ccc} \Pi_{n+1}(G/PL) & \xrightarrow{\partial_n} & \Pi_n(PL/O) \\ \Big\downarrow{\scriptstyle\circ\Sigma\alpha} & & \Big\downarrow{\scriptstyle\circ\alpha} \\ \Pi_{n+k+1}(G/PL) & \xrightarrow{\partial_{n+k}} & \Pi_{n+k}(PL/O) \end{array} \tag{2.1}$$

The commutativity of (2.1) is a consequence of a well-known formula in fibre-space theory. From [4], we know that bP_{i+1} can be described homotopy-theoretically as the image of $\partial_i:\Pi_{i+1}(G/PL)\to\Pi_i(PL/O)$, so to prove the theorem, it is sufficient to prove the left-hand vertical arrow in (2.1) is the zero map.[5]

To this end, observe first that since $k>0$, $\Sigma\alpha$ has finite order. We may further clearly assume $\Sigma\alpha$ has prime-power order p^j and distinguish two cases, according as p is odd or $p=2$.

For p odd, the conclusion is trivial because the only torsion in $\Pi_*(G/PL)$ is of order 2 [4]. If $p=2$, we argue as follows. Let Y be the space

$$(K(Z_2,2)\times_{\delta Sq^2}K(Z,4))\times\Pi_{i\geqq 2}K(Z_2,4i-2)\times K(Z,4i).$$

Sullivan [4] shows that after localizing at the prime 2, Y and G/PL become homotopy equivalent. We are therefore reduced to proving that $\Pi_{n+1}(Y)\xrightarrow{\circ\Sigma\alpha}\Pi_{n+k+1}(Y)$ is the zero map, which is evident from the structure of Y.

[4] See Rourke-Sanderson, *Block bundles*. III, Ann. of Math. (2) **87** (1968), 431–483, for a homotopy-theoretic description of the set $\Gamma^f_{i+l,i}$ and the maps φ_i, ω_i.

[5] It is actually true that the composition pairing $\Pi_{n+1}(G/PL)\times\Pi_{n+k+1}(S^{n+1})\to\Pi_{n+k+1}(G/PL)$ is trivial but we do not need this additional fact.

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