ON A CONSTRUCTION OF BREDON
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Abstract. Using a homotopy-theoretical description of a geometric pairing due to Bredon, we show how to rederive Bredon's results on the pairing. Furthermore, we are able to, in some sense, complete these results by combining the homotopy-theoretical approach with Sullivan's determination of the 2-primary Postnikov decomposition of the space $G/PL$.

1. Introduction. In [1], Bredon introduced a geometric pairing

$$\Gamma_n \times \Pi_{n+k}(S^n) \xrightarrow{\rho_{n,k}} \Gamma_{n+k},$$

$\Gamma_i$ being the group of differential structures on the $i$-sphere and used this pairing to construct certain semifree actions of groups on spheres. There is also the pairing

$$\Pi_n(PL/O) \times \Pi_{n+k}(S^n) \xrightarrow{\gamma_{n,k}} \Pi_{n+k}(PL/O)$$

defined by composition, $PL/O$ being the fibre of the natural map $BO \to BPL$ of classifying spaces. Since, by smoothing theory, $\Pi_n(PL/O)$ and $\Gamma_n$ are isomorphic groups, it is natural to inquire about the relation between these two pairings. In fact, as observed by Bredon [1] and the present author [3], these two pairings coincide.

While Bredon's geometric definition is rather natural and simple, the main results and proofs are more perspicuous using the homotopy-theoretical definition. Thus, in §2 we show how to retrieve (and slightly generalize) the theorems of [1, §§1, 2] in this context, in particular the following important result, which we state as

**Theorem A** [1, Theorem 2.1]. Let $\alpha \in \Pi_{n+k}(S^n)$, $\sigma \in \Gamma_n$ and let $p_1(\sigma) \subset \Pi_{n+1}(S)$ denote the set of all elements represented (via the Pontryagin-Thom construction) by framed embeddings $\Sigma^n \times D^1 \subset S^{n+1}$, $\Sigma^n$...
a homotopy sphere representing $\sigma$. Then

$$p_i(\sigma) \circ \Sigma^i \alpha \subseteq p_i(\rho_{n, k}(\sigma), \alpha),$$

where $\Sigma^i \alpha$ is the $i$th iterated suspension of $\alpha$.

However, the main contribution of this note is the following Theorem B (essentially conjectured by Bredon [1, Corollary 2.3 and succeeding remarks]) which, taken together with Theorem A, gives a rather good hold on Bredon's pairing.

**Theorem B.** If $bP_{n+1}$ is the subset of $\Gamma_n$ consisting of those homotopy spheres which bound $\Omega$-manifolds, then the pairing $\rho_{n, k}$ restricts to a pairing $\tilde{\rho}_{n, k}: bP_{n+1} \times \Pi_{n+k}(S^n) \to bP_{n+k+1}$. Moreover, $\tilde{\rho}_{n, k}$ is trivial for $k > 0$.

The proof of Theorem B, which relies on results of Sullivan [4], will also be carried out in §2.

As a final introductory comment, we remark that while the main interest in the composition pairing $\Pi_n(PL/O) \times \Pi_{n+k}(S^n) \to \Pi_{n+k}(PL/O)$ lies in its geometric interpretation, the pairing has also proved to be of some use in studying the $k$-invariants of $PL/O$ ([5], [3]).

2. The main properties of $\rho_{n, k}$. In this section we deduce the main properties of the pairing $\rho_{n, k}$, working in the homotopy-theoretical context. We begin with a proposition which summarizes and slightly generalizes the results of [1, §1].

**Proposition.** The pairing $\rho_{n, k}: \Gamma_n \times \Pi_{n+k}(S^n) \to \Gamma_{n+k}$ is bilinear and associative in the sense that the diagram

\[
\begin{array}{ccc}
\Gamma_n \times \Pi_{n+k}(S^n) \times \Pi_{n+k+1}(S^{n+k}) & \xrightarrow{id \times \text{comp}} & \Gamma_n \times \Pi_{n+k+1}(S^n) \\
\downarrow \rho_{n, k} \times id & & \downarrow \rho_{n, k+1} \\
\Gamma_{n+k} \times \Pi_{n+k+1}(S^{n+k}) & \xrightarrow{\rho_{n+k, l}} & \Gamma_{n+k+1}
\end{array}
\]

is commutative.

**Proof.** Using the identification of $\rho_{n, k}$ with $\gamma_{n, k}$, all these statements, except perhaps the linearity in the first variable, are trivial. As for linearity in the first variable, this follows from the fact that $PL/O$ is an $H$-space.\(^3\)

We remark that linearity in the first variable was proved in [1] only under the assumption that $\Pi_{n+k}(S^n)$ is stable, i.e. $k < n-1$. Our proof, exploiting the $H$-structure on $PL/O$, shows this restriction on $k$ to be unnecessary.

\(^3\) The bilinearity of $\gamma_{n, k}$ has been independently observed by Schultz (Smooth structures on $S^p \times S^q$, Ann. of Math. (2) 90 (1969), 187-198) in a closely related context.
We come now to the two key results, Theorems A and B.

**Proof of Theorem A.** Let $\Gamma_{n+1}$ denote the set of framed embeddings of homotopy $n$-spheres in $S^{n+1}$, $G_i$ the space of maps $S^{i+1} \to S^{i+1}$ of degree 1. In addition to the obvious (forgetful) map $\varphi_i : \Gamma_{i+1} \to \Gamma_i$, there is a map $\omega_i : \Gamma_{i+1} \to \Pi_i(G_i)$ described in [2]. It is not difficult to see that the set $p_i(\sigma)$, as originally defined by Kervaire-Milnor, can equivalently be described as $J_n(\omega_n(p_{n+1}^\sigma))$, where $J_n : \Pi_n(G_i) \to \Pi_{n+1}(S^n)$ is obtained, as usual, by the Hopf construction. Thus, if $\beta \in p_i(\sigma)$, there exists $\eta \in \omega_n(q_n^\sigma(\sigma))$ such that $\beta = J_n(\eta)$. But by a known formula (cf. [1, p. 442]) we have $J_n(\eta) \circ \Sigma \alpha = J_{n+k}(\eta \circ \alpha)$. Moreover, it is not difficult to see, for example by using the homotopy-theoretical interpretation of $\Gamma_{i+1}$, $\varphi_i$, $\omega_i$ (see footnote 4) that $\eta \circ \alpha \in \omega_{n+k}(q_{n+1}^\sigma(\rho_{n+k}(\sigma, \alpha)))$ and the theorem follows.

**Proof of Theorem B.** If $G_k$ is the set of maps $S^k \to S^k$ of degree $\pm 1$ and $G = \lim_{k \to \infty} G_k$, there are natural maps $\text{O} \to PL \to G$ and a fibration $PL/O \to G/O \to G/PL$. For $\alpha \in \Pi_{n+k}(S^n)$, we consider the diagram.

$$
\begin{array}{ccc}
\Pi_{n+1}(G/PL) & \xrightarrow{\partial_n} & \Pi_n(PL/O) \\
\downarrow \circ \Sigma \alpha & & \downarrow \circ \alpha \\
\Pi_{n+k+1}(G/PL) & \xrightarrow{\partial_{n+k}} & \Pi_{n+k}(PL/O)
\end{array}
$$

(2.1)

The commutativity of (2.1) is a consequence of a well-known formula in fibre-space theory. From [4], we know that $b_{PL}$ can be described homotopy-theoretically as the image of $\partial : \Pi_{i+1}(G/PL) \to \Pi_i(PL/O)$, so to prove the theorem, it is sufficient to prove the left-hand vertical arrow in (2.1) is the zero map.\(^{5}\)

To this end, observe first that since $k > 0$, $\Sigma \alpha$ has finite order. We may further clearly assume $\Sigma \alpha$ has prime-power order $p^j$ and distinguish two cases, according as $p$ is odd or $p = 2$.

For $p$ odd, the conclusion is trivial because the only torsion in $\Pi_{n+k}(G/PL)$ is of order 2 [4]. If $p = 2$, we argue as follows. Let $Y$ be the space

$$(K(\mathbb{Z}_2, 2) \times_{\delta S^4} K(\mathbb{Z}, 4)) \times \Pi_{i \geq 2} K(\mathbb{Z}, 4i - 2) \times K(\mathbb{Z}, 4i).$$

Sullivan [4] shows that after localizing at the prime 2, $Y$ and $G/PL$ become homotopy equivalent. We are therefore reduced to proving that $\Pi_{n+1}(Y) \Sigma_{\equiv} \Pi_{n+k+1}(Y)$ is the zero map, which is evident from the structure of $Y$.

\(^4\) See Rourke-Sanderson, *Block bundles*. III, Ann. of Math. (2) 87 (1968), 431–483, for a homotopy-theoretic description of the set $\Gamma_{i+1}$ and the maps $\varphi_i$, $\omega_i$.

\(^5\) It is actually true that the composition pairing $\Pi_{n+1}(G/PL) \times \Pi_{n+k+1}(S^{n+1}) \to \Pi_{n+k+1}(G/PL)$ is trivial but we do not need this additional fact.
BIBLIOGRAPHY


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