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NORMABILITY OF CERTAIN TOPOLOGICAL RINGS

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ABSTRACT. Criteria are given for the normability of certain topological rings. These criteria yield generalizations of known theorems on the normability of compact integral domains and topological fields.

In [6, Theorem 7], Z. S. Lipkina proved that the topology of a metrizable, compact ring A with identity and without zero-divisors is given by a norm, i.e., a function $\| \ \|$ from A into the nonnegative real numbers satisfying $\|x\|=0$ if and only if $x=0$, $\|-x\|=\|x\|$, $\|x+y\|\leqq\|x\|+\|y\|$, and $\|xy\|\leqq\|x\|\ \|y\|$ for all $x, y\in A$. Her proof depended upon exhibiting A as a quotient ring of a certain power series ring in countably many noncommuting variables over a coefficient ring whose elements did not, however, commute with the variables. Here we shall give an elementary proof of a generalization of her result concerning normability.

A function v defined on a ring A is a *natural pseudovaluation* if the range of v is contained in the natural numbers together with $+\infty$ and if for all $x, y\in A$, $v(x)=+\infty$ if and only if $x=0$, $v(-x)=v(x)$, $v(x+y)\geqq \min\{v(x), v(y)\}$, and $v(xy)\geqq v(x)+v(y)$. Clearly the exponential of a natural pseudovaluation to a base <1 is a nonarchimedean norm satisfying $\|x\|\leqq 1$ for all $x\in A$. An ideal $\mathfrak{a}$ in a topological ring is *topologically nilpotent* if the filter base formed by its powers converges to zero.

THEOREM 1. *The topology of a topological ring A is defined by a natural pseudovaluation if and only if the following three conditions hold*:

1° *A is metrizable.*

2° *The open ideals of A form a fundamental system of neighborhoods of zero.*

3° *A contains an open, topologically nilpotent ideal.*

PROOF. *Necessity.* Let $\mathfrak{a}_n=\{x\in A: v(x)\geqq n\}$. Then $\{\mathfrak{a}_n\}_{n\geqq 1}$ is a fundamental system of neighborhoods of zero, each $\mathfrak{a}_n$ is an ideal, and $\mathfrak{a}_1^n\subseteq\mathfrak{a}_n$ for all $n\geqq 1$.

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Sufficiency. Let $\mathfrak{r}$ be an open, topologically nilpotent ideal; by 1° and 2° there is a fundamental decreasing sequence $\{\mathfrak{a}_k\}_{k\geqq 1}$ of ideal neighborhoods of zero each contained in $\mathfrak{r}$. For each $n\geqq 1$ let $\mathfrak{b}_n=\sum \mathfrak{a}_{k_1}\mathfrak{a}_{k_2}\cdots\mathfrak{a}_{k_r}$, the sum extending over all r-tuples $(k_1, k_2, \cdots, k_r)$ such that $1\leqq r\leqq n$, $1\leqq k_i\leqq n$ for all $i\in[1, r]$, and $k_1+k_2+\cdots+k_r=n$. Since $\mathfrak{a}_n$ is one of those summands, $\mathfrak{a}_n\subseteq\mathfrak{b}_n$. Thus $\mathfrak{b}_n$ is an open ideal of A. Clearly $\mathfrak{b}_n\mathfrak{b}_m\subseteq\mathfrak{b}_{n+m}$ for all $n, m\geqq 1$. If $n\geqq m$, then $\mathfrak{b}_n\subseteq\mathfrak{b}_m$, for if $k_1+\cdots+k_r=n$, we may find, for some $s\leqq r$, positive integers $j_1, \cdots, j_s$ such that $j_i\leqq k_i$ for all $i\in[1, s]$ and $j_1+\cdots+j_s=m$, whence $\mathfrak{a}_{k_1}\cdots\mathfrak{a}_{k_r}\subseteq\mathfrak{a}_{j_1}\cdots\mathfrak{a}_{j_s}\subseteq\mathfrak{b}_m$. We shall next show that $\mathfrak{b}_{n^2}\subseteq\mathfrak{r}^n+\mathfrak{a}_{n+1}$. Let $k_1+\cdots+k_r=n^2$. If each $k_i\leqq n$, then $n^2=k_1+\cdots+k_r\leqq nr$, so $n\leqq r$, whence $\mathfrak{a}_{k_1}\cdots\mathfrak{a}_{k_r}\subseteq\mathfrak{r}^r\subseteq\mathfrak{r}^n$. If $k_i\geqq n+1$ for some i, then $\mathfrak{a}_{k_1}\cdots\mathfrak{a}_{k_r}\subseteq\mathfrak{a}_{k_i}\subseteq\mathfrak{a}_{n+1}$. Therefore $\mathfrak{b}_{n^2}\subseteq\mathfrak{r}^n+\mathfrak{a}_{n+1}$. Given m, let n be so large that $n+1\geqq m$ and $\mathfrak{r}^n\subseteq\mathfrak{a}_m$; then $\mathfrak{b}_{n^2}\subseteq\mathfrak{a}_m$. Hence $\{\mathfrak{b}_n\}_{n\geqq 1}$ is a fundamental system of ideal neighborhoods of zero of A satisfying $\mathfrak{b}_n\mathfrak{b}_m\subseteq\mathfrak{b}_{n+m}$ for all $n, m\geqq 1$. Therefore the function v defined by $v(x)=0$ if $x\notin\mathfrak{b}_1$, $v(x)=n$ if $x\in b_n$ but $x\notin\mathfrak{b}_{n+1}$, $v(0)=+\infty$ is a natural pseudovaluation defining the topology of A.

A topologically artinian ring is a Hausdorff topological ring A such that the open left ideals of A form a fundamental system of neighborhoods of zero and for every open left ideal $\mathfrak{a}$, the left A-module $A/\mathfrak{a}$ is artinian (it is not necessary to assume here that A has an identity element). The complete topologically artinian rings are precisely the strictly linearly compact rings [**2**, p. 111] (called "im engeren Sinne linear kompakt" rings in [**5**]).

THEOREM 2. *If A is a topologically artinian ring and if every open left ideal contains an open ideal, then the radical $\mathfrak{r}$ of A is topologically nilpotent.*

PROOF. Let $\mathfrak{b}$ be an open ideal of A; then $A/\mathfrak{b}$ is an artinian ring and hence has a nilpotent radical. Since $(\mathfrak{r}+\mathfrak{b})/\mathfrak{b}$ is contained in the radical of $A/\mathfrak{b}$, we conclude that $\mathfrak{r}^n\subseteq\mathfrak{b}$ for some $n\geqq 1$. Thus $\mathfrak{r}$ is topologically nilpotent.

A norm is *bounded* if its values are contained in a bounded interval.

THEOREM 3. *Let A be a strictly linearly compact ring, and let $\mathfrak{r}$ be the radical of A. The following statements are equivalent:*

1° *The topology of A is defined by a bounded, nonarchimedean norm.*

2° *The topology of A is defined by a norm, and every open left ideal contains an open ideal.*

3° *A is metrizable, $\mathfrak{r}$ is open, and every open left ideal contains an open ideal.*

4° *The topology of A is defined by a natural pseudovaluation.*

PROOF. For a given norm, let B_s denote the closed ball of center 0 and radius s. To show that 1° implies 2°, let $\|\ \|$ be a nonarchimedean

norm satisfying $\|x\| \leqq c$ for all $x \in A$, where $c \geqq 1$. Then for any $t > 0$, the ideal generated by B_{t/c^2} is contained in B_t, since $\|ax\| \leqq t/c$, $\|xb\| \leqq t/c$, and $\|axb\| \leqq t$ for all $x \in B_{t/c^2}$ and all a, $b \in A$. Thus 2° holds.

If 2° holds, then $B_{1/2}$ contains an open ideal $\mathfrak{n}$; clearly $\lim x^n = 0$ for all $x \in \mathfrak{n}$, so $\mathfrak{n} \subseteq \mathfrak{r}$ by [**4**, Corollary, Theorem 12] as A is complete, whence 3° holds. By Theorems 1 and 2, 3° implies 4°, and clearly 4° implies 1°.

COROLLARY 1. *Conditions* 1°, 2°, *and* 4° *of Theorem* 3 *are equivalent for a topologically artinian ring.*

PROOF. We need only apply Theorem 3 to the completion of a topologically artinian ring, since the completion is clearly strictly linearly compact.

COROLLARY 2 *Let A be a totally disconnected compact ring, and let $\mathfrak{r}$ be the radical of A. The following statements are equivalent*:

1° *The topology of A is defined by a norm.*

2° *A is metrizable and $\mathfrak{r}$ is open.*

3° *The topology of A is defined by a natural pseudovaluation.*

PROOF. The open ideals of A form a fundamental system of neighborhoods of zero [**4**, Lemma 9], so A is strictly linearly compact since $A/\mathfrak{b}$ is a finite ring for any open ideal $\mathfrak{b}$.

A compact ring with identity is totally disconnected, [**4**, Theorem 8] and if it has no divisors of zero, its radical is open [**4**, Theorem 19]. Hence Corollary 2 generalizes [**6**, Theorem 7].

Here is an example of a commutative, metrizable, linearly compact ring with open radical whose topology is not given by a norm: Let $A = K[[X, Y]]$, the ring of formal power series in two variables over a field K. Since A is a local noetherian domain complete for the topology defined by the powers of its radical $\mathfrak{r} = (X) + (Y)$, A is linearly compact for the discrete topology [**7**, pp. 271–272] and hence for any weaker, Hausdorff linear topology (cf. [**2**, Exercises 14–20, pp. 108–111], [**5**]). Moreover, A is a unique factorization domain [**7**, Theorem 6, p. 148]. Let $(p_n)_{n \geqq 1}$ be a sequence of irreducible elements no two of which are associates (e.g., let $p_n = X + Y^n$), let $\mathfrak{a}_n$ be the principal ideal generated by $p_1 \cdots p_n$, and let $\mathscr{T}$ be the topology on A for which $\{\mathfrak{a}_n\}_{n \geqq 1}$ is a fundamental system of neighborhoods of zero. As A is a unique factorization domain, $\bigcap_{n \geqq 1} \mathfrak{a}_n = (0)$; as A is local, $\mathfrak{a}_1 \subseteq \mathfrak{r}$. Equipped with topology $\mathscr{T}$, therefore, A is a commutative, linearly compact, metrizable ring with open radical. If $\mathscr{T}$ were defined by a norm, then $\{x \in A : \lim x^n = 0\}$ would be a neighborhood of zero and hence would contain $\mathfrak{a}_m$ for some m; therefore $\lim_n (p_1 \cdots p_m)^n = 0$, and in particular, $(p_1 \cdots p_m)^n \in \mathfrak{a}_{m+1}$ for

some n, whence p_{m+1} would divide $p_1^n \cdots p_m^n$, a contradiction of the fact that A is a unique factorization domain.

We note finally that P. M. Cohn's criterion for the normability of a topological field may be extended to topological rings. A subset B of a topological ring is *bounded* if for every neighborhood V of zero there is a neighborhood U of zero such that $UB \subseteq V$ and $BU \subseteq V$. Our proof of the following theorem is a simplification of proofs in [3].

THEOREM 4. *Let A be a Hausdorff topological ring whose center contains a cancellable element a such that $x \mapsto ax$ is an open mapping and* $\lim a^n = 0$. *If A has a bounded neighborhood of zero* [*a bounded, open additive subgroup*], *then the topology of A is defined by a norm* [*a nonarchimedean norm*].

PROOF. Let A_1 be the rings of all the fractions x/y, where $x \in A$ and where y is a cancellable element of A belonging to the center of A such that $z \mapsto zy$ is an open mapping, topologized by declaring the neighborhoods of zero in A a fundamental system of neighborhoods of zero in A_1. Clearly A_1 is a topological ring containing A as an open subring; consequently, replacing A with A_1, if necessary, we may assume that a is invertible in A.

Let V be a bounded neighborhood of zero. Replacing V by $V \cap (-V)$, if necessary, we may assume that V is symmetric. Let $U = \{x \in A : xV \subseteq V\}$. As V is symmetric and bounded, U is a symmetric neighborhood of zero. To show that U is bounded, let W be a neighborhood of zero. As V is bounded, there exists a neighborhood T of zero such that $TV \subseteq W$, $VT \subseteq W$; as $\lim a^n = 0$, $a^k \in V$ for some k; hence $a^k T$ is a neighborhood of zero, and $Ua^k T \subseteq VT \subseteq W$, and similarly $a^k TU = TUa^k \subseteq W$. Thus U is a bounded symmetric neighborhood of zero that clearly satisfies $UU \subseteq U$. Let U_1 be a neighborhood of zero such that $U_1 \subseteq U$ and $U_1 U + U_1 U + U_1 U \subseteq U$; replacing a by a power of a, if necessary, we may assume that $a \in U_1$; then $aU + aU + aU \subseteq U$. Clearly $(a^k U)_{k \in \mathbf{Z}}$ is a decreasing sequence of neighborhoods of zero; it is a fundamental system of neighborhoods of zero, for if Y is a neighborhood of zero, there exists a neighborhood Z of zero such that $ZU \subseteq Y$ as U is bounded, and there exists $t \geqq 0$ such that $a^t \in Z$ since $\lim a^n = 0$, whence $a^t U \subseteq ZU \subseteq Y$. In particular, $\bigcap_{k \geqq 0} a^k U = (0)$.

Let $g(0) = 0$, and for each nonzero $x \in A$ let $g(x) = 2^{-k}$, where k is the largest of the integers j such that $x \in a^j U$ (such an integer exists, for as $\lim a^n x = 0$, $x \in a^{-n} U$ for some $n \geqq 0$). If V is an additive subgroup, then U is a subring, and g is clearly a nonarchimedean norm defining the topology of A.

In general, let $f(x) = \inf\{\sum_{i=1}^p g(z_i) : z_1 + \cdots + z_p = x\}$. Since $aU + aU + aU \subseteq U$, $a^{n+1}U + a^{n+1}U + a^{n+1}U \subseteq a^n U$ for all $n \in \mathbf{Z}$; by the argument

of [1, Proposition 2, p. 15], f is a norm defining the topology of the additive group A (by induction on p, one shows that $\frac{1}{2}g(\sum_{i=1}^{p} z_i) \leqq \sum_{i=1}^{p} g(z_i)$). Since $a \in U$ and $UU \subseteq U$, $g(xy) \leqq g(x)g(y)$ for all x, $y \in A$, whence $f(xy) \leqq f(x)f(y)$; thus f is a norm defining the topology of the ring A.

COROLLARY [3, THEOREMS 6.1 AND 7.1]. *The topology of an indiscrete, Hausdorff topological field K is defined by a norm [a nonarchimedean norm] if and only if there is a nonzero element a of K such that* $\lim a^n = 0$ *and K contains a bounded neighborhood of zero [a bounded, open additive subgroup].*

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