PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

# SPLITTING IN MAP GROUPS

LEWIS ROBERTSON AND THEODORE W. WILCOX

ABSTRACT. Every locally compact maximally almost periodic group $G$ has a normal vector subgroup, the centralizer of which is of finite index. This vector subgroup is nontrivial whenever the identity component of $G$ is not compact. Furthermore, if $G$ has relatively compact conjugacy classes, then $G \cong \boldsymbol{R}^n \times L$ where $L$ has a compact open normal subgroup. Several structure theorems are also obtained for cases in which splitting need not occur.

Recall that a locally compact maximally almost periodic, [MAP], group is one which has sufficiently many continuous finite-dimensional unitary representations to separate points. The classical theorem upon which this work is based is the following:

THEOREM (FREUDENTHAL-WEIL). *Let G be a connected locally compact group. Then G is MAP if and only if G is a direct product of a vector group and a compact group.*

In §1, it is shown that a [MAP]-group $G$ has a normal vector subgroup, the centralizer of which has finite index in $G$. Furthermore, a general splitting theorem of the Freudenthal-Weil type is proved. In §2, some structural results are obtained for [MAP]-groups in which the vector group need not split.

The following notation for certain classes of topological groups was established in [**2**].

IN—the groups contain a compact neighborhood of $e$ invariant under the action of the inner automorphisms.

SIN—(read "small invariant neighborhoods") the groups have a basic system at $e$ of compact invariant neighborhoods.

FC—the groups have finite conjugacy classes.

$[FC]^-_{\mathscr{B}}$—the groups are locally compact and have relatively compact $\mathscr{B}$-conjugacy classes, where $\mathscr{B}$ is a subgroup of the automorphism group. That is, $G \in [FC]^-_{\mathscr{B}}$ means $\{x \in G : \exists \beta \in \mathscr{B}, \beta x = y\}^-$ is compact for each $y \in G$.

---

Received by the editors January 4, 1971.

*AMS* 1970 *subject classifications*. Primary 22D05, 22D10.

*Key words and phrases*. Maximally almost periodic, finite-dimensional unitary representation, MAP-groups, $[FC]^-$-groups, splitting.

In general, the use of brackets in this notation indicates that the groups considered are locally compact and a subscript indicates that the notion is being modified relative to some subgroup of the topological automorphism group.

Finally, we denote the group of topological automorphisms of $G$ by $\mathfrak{A}(G)$ and the connected component of the identity of $G$ by $G_0$. If $G$ is a locally compact abelian group, then $\hat{G}$ is the character group of $G$. If $H \subset G$ is a subgroup, then $H^{\perp} = \{\chi \in \hat{G} : \chi(h) = 1, \text{ all } h \in H\}$ is the annihilator of $H$ in $\hat{G}$. All groups considered will be Hausdorff.

1. Let $G$ be a locally compact abelian group and let $\mathfrak{B}$ be a subgroup of $\mathfrak{A}(G)$. For each $\beta \in \mathfrak{B}$ and $\chi \in \hat{G}$, define $\beta^*(\chi) = \chi \circ \beta^{-1} \in \hat{G}$. Let $\mathfrak{B}^* = \{\beta^* : \beta \in \mathfrak{B}\}$, so that $\mathfrak{B}^* \subset \mathfrak{A}(\hat{G})$ and, in fact, it is well known (compare [**3**, Theorem 26.9]) that the map $\beta \to \beta^*$ is a topological isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}^*$. By $F(\hat{G}, \mathfrak{B}^*)$ we denote the set of characters $\chi$ in $\hat{G}$ for which $\{\beta^*\chi : \beta^* \in \mathfrak{B}^*\}$ is finite. We note that this is a slight change from the notation of [**8**]. It is easy to see that $F(\hat{G}, \mathfrak{B}^*)$ is a subgroup of $\hat{G}$. Furthermore, a subgroup $L$ of $\hat{G}$ separates points in $G$ if and only if $L$ is dense in $\hat{G}$. Thus the following formulation of [**8**, Theorem 1] is valid.

Proposition 1. *Let $H$ be a normal locally compact abelian subgroup of a topological group $G$. Let $\mathfrak{B}$ be the subgroup of the group of topological automorphisms of $H$ consisting of the restrictions to $H$ of inner automorphisms of $G$. Then $F(\hat{H}, \mathfrak{B}^*)$ is a dense subgroup of $\hat{H}$ if $G \in MAP$.*

The following theorem is a variant of a result which was previously announced [**7**, Theorem 4]. We note that in Proposition 1 and Theorem 1, it is not necessary to assume that $G$ be locally compact.

Theorem 1. *Let $G \in MAP$ and let $H$ be a normal subgroup of $G$ such that $H = R^n \times T^m$ where $n$ and $m$ are nonnegative integers. Then the centralizer, $C(H)$, of $H$ has finite index in $G$.*

Proof. We identify $\hat{H}$ with $R^n \times Z^m$ so that, in the notation of Proposition 1, $F(\hat{H}, \mathfrak{B}^*)$ is a dense subgroup of $R^n \times Z^m$. Each topological automorphism $\beta^*$ in $\mathfrak{B}^*$ can be extended linearly to a continuous homomorphism $\beta^{\#}$ of $R^{n+m}$. Furthermore, since the rationals are dense in the reals, each $\beta^{\#}$ is a linear operator on $R^{n+m}$. Let $\mathfrak{B}^{\#}$ be the collection of linear operators so obtained. Clearly the points of $R^{n+m}$ which are finitely orbited by $\mathfrak{B}^{\#}$ form a dense subset of $R^{n+m}$ and hence must contain a basis. If $x_j$ is an element of this basis, then, since the set $\{\beta^{\#}x_j : \beta^{\#} \in \mathfrak{B}^{\#}\}$ is finite, there can only be finitely many $j$th columns in the matrix representations of the $\beta^{\#}$. It follows that $\mathfrak{B}^{\#}$, hence $\mathfrak{B}^*$ and $\mathfrak{B}$ are all finite. On the other hand, the cardinality of $\mathfrak{B}$ equals $[G : C(H)]$. That is, the centralizer of $H$ in $G$ has finite index in $G$.

We wish to prove that there are normal vector subgroups in locally compact MAP groups and for this purpose we require the following:

LEMMA 1. *Let $G$ be a locally compact abelian group which is the direct product of an $n$-dimensional vector group $N$ and a compact group $C$. Suppose that $\{E_i : i\in I\}$ is a descending chain of closed subgroups such that $E_i/(E_i\cap C)$ is an $n$-dimensional vector group. Let $E=\bigcap\{E_i : i\in I\}$. Then $E/(E\cap C)$ is an $n$-dimensional vector group.*

PROOF. This lemma was first proved in [**4**, 9.3 Lemma, p. 34]. The following is a shorter and simpler proof. Since $G$ is the direct product of $N$ and $C$, we have $\hat{G}=C^{\perp}N^{\perp}$, where $C^{\perp}\cong R^n$ and $C^{\perp}$ is open in $\hat{G}$. Note that $C$ being compact implies that $E_iC/C\cong E_i/(E_i\cap C)\cong \boldsymbol{R}^n$. It follows that $G=E_iC$. That is, $E_i^{\perp}\cap C^{\perp}=\{e\}$ so that $E_i^{\perp}\cap(C^{\perp}-\{e\})=\varnothing$. Thus $E^{\perp}=(\bigcup E_i^{\perp})^{-}$ does not meet the open set $C^{\perp}-\{e\}$. Consequently, $E^{\perp}\cap C^{\perp}=\{e\}$ and $G=EC$. Thus $\boldsymbol{R}^n\cong G/C=EC/C\cong E/(E\cap C)$.

THEOREM 2. *Let $G\in[MAP]$. Then $G$ contains a normal vector subgroup $V$ of dimension $n$. This $n$ equals the dimension of the vector component of the connected component of the identity of $G$.*

PROOF. Let $H$ be the connected component of the center of the connected component of $G$. That is, $G\supset G_0\supset Z(G_0)\supset (Z(G_0))_0=H$. Since $H$ is a connected locally compact abelian group, $H=NC$ is the direct product of an $n$-dimensional vector group $N$ and a compact connected group $C$. Furthermore, it is easy to see that this $n$ equals the $n$ in the statement of the theorem and that $C$ is the unique maximal compact subgroup of $H$. Let $\mathfrak{C}$ be the collection of subgroups $E$ of $H$ such that

1. $E$ is a closed normal subgroup of $G$ and
2. $E/(E\cap C)$ is an $n$-dimensional vector group.

We note that $\mathfrak{C}$ is nonempty since $H$ is a characteristic subgroup of $G$ and consequently $H\in\mathfrak{C}$. By Zorn's lemma and Lemma 1, there is a minimal element $E$ in $\mathfrak{C}$ where $E=VA$ is a direct product of an $n$-dimensional vector group $V$ and a compact abelian group $A$. Furthermore, $A$ is connected for otherwise $E_0=VA_0$ would be a proper characteristic subgroup of $E$ and the quotient of $E_0$ by the subgroup $A_0=E_0\cap C$ of compact elements of $E_0$ would be an $n$-dimensional vector group which would contradict the minimality of $E$ in $\mathfrak{C}$.

If $A=\{e\}$, we are done. Thus we assume that $A$ is nontrivial and will arrive at a contradiction. Since $G\in$[MAP], there is a representation $\Pi$ of $G$, that is, a continuous homomorphism $\Pi: G\rightarrow U$ where $U$ is a (unitary) Lie group, such that $\Pi(A)\neq\{I\}$. Let $K=A\cap\ker\Pi$. Then $K$ is a normal subgroup of $G$ since $A$ is a characteristic subgroup of the normal subgroup $E$. Furthermore, $A/K\cong\Pi(A)$ is a connected compact abelian Lie group;

that is, $A/K \cong T^m$ for some positive integer $m$. It follows that $E/K$ is a normal subgroup of $G/K$ and that $E/K \cong R^n \times T^m$. Since $G \in$[MAP] and $K$ is compact, we have $G/K \in$[MAP]. (See [**5**, Proposition 1].) By Theorem 1, the centralizer of $E/K$ has finite index in $G/K$. Thus the group of automorphisms of $E/K$ which are the restrictions to $E/K$ of inner automorphisms of $G/K$ is finite. By [**1**, Lemma 1] or [**4**, Theorem X] there is a closed subgroup $W/K$ of $E/K$ with $W/K \cong R^n$ which is invariant under the action of this finite group of automorphisms. It follows that $W$ is a proper subgroup of $E$, $W$ is a normal subgroup of $G$, and $K = W \cap C$. Thus $W \in \mathfrak{C}$ which contradicts the choice of $E$.

THEOREM 3. *Let* $G \in [MAP] \cap [FC]^-$. *Then* $G \in [SIN]$. *Furthermore,* $G \cong V \times H$ *where* $V$ *is a vector group,* $H \in [FC]^- \cap [MAP]$, *and* $H$ *contains a compact open normal subgroup* $K$.

PROOF. Let $P$ be the set of those elements of $G$ which are contained in compact normal subgroups of $G$. By the structure theorem for $[FC]^-$-groups [**2**, Theorem 3.16], we know that $P$ is a closed normal subgroup of $G$ and that $G/P$ is a direct product of a vector group and a discrete group $D$. Let $V$ be the normal vector subgroup of $G$ obtained in Theorem 2. Since the connected component of the identity in $G$ is a direct product of $V$ with a compact group, it follows that the connected component, $(G/P)_0$, of $G/P$ is $VP/P$. Thus we have the following composition of maps

$$G \xrightarrow{\pi_1} G/P = VP/P \times D \xrightarrow{\pi_2} VP/P \xrightarrow{\varphi} V$$

where $\varphi$ is a topological isomorphism, $\varphi(vP) = v$ for $v \in V$. Clearly the above composition is a continuous projection of $G$ onto $V$ so that $G \cong V \times H$. It is known that every $[FC]^-$ group has an invariant neighborhood. (For a proof see Corollary 2.2 of *Dual spaces of locally compact groups with precompact conjugacy classes* by J. Liukkonen which will soon appear in Trans. Amer. Math. Soc.) That is, $G \in$[IN]. Since [MAP]$\cap$[IN]$\subset$[SIN] (see [**4**, Proposition 12.2]), $G$ has small invariant neighborhoods and it follows that $H$ contains a compact open normal subgroup $K$.

2. The material in this section is motivated by the following example. Let $G$ be the group of all matrices

$$\begin{pmatrix} 1 & n & x \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}, \qquad n, m \in Z, x \in R$$

and let $H$ be the copy of the real line in $G$ consisting of the above matrices when $n = m = 0$. Then $G$ is a compactly generated [MAP] group (thus also in [SIN]), $H$ is the connected component, the center and the commutator subgroup of $G$, and $G/H \cong Z \times Z$ but $H$ does not split in $G$. This is essentially Example 11.10 of [**4**]. We ask about the extent that this example is typical of the situation when splitting does not occur.

The following theorem is a modification of [**2**, Proposition 4.5].

THEOREM 4. *Let $G$ be a compactly generated group in [MAP]. Assume that $G/G_0$ is in $[FC]^-$. Then $G$ contains an open normal subgroup $H$ such that*

(1) *$H = VK$ is a semidirect product of a vector group $V$ normal in $G$ and a compact group $K$ and*

(2) *$G/H \cong Z^n$ for some nonnegative integer $n$.*

PROOF. We recall that a compactly generated [MAP] group is in [SIN] [**4**, Proposition 12.2 (iv)]. Since $G \in [\mathrm{MAP}] \cap [\mathrm{SIN}]$, by Theorem 2, $G$ contains a normal vector group $V$ and an open normal subgroup $N$ such that $N/V$ is compact. By Iwasawa's theorem (see e.g., [**4**, Proposition 7.6]) $N = VC$ is a semidirect product of $V$ with a compact group $C$. Then $G^* = G/N$ is a finitely generated (discrete) FC group. By [**6**, Theorem 5.1] the commutator subgroup of $G^*$ is finite so that $G^*$ contains a finite normal subgroup $F^*$ such that $G^*/F^* \cong Z^n$. Let $\pi: G \to G/N = G^*$ be the natural projection and let $H = \pi^{-1}(F^*)$. Since $H/N \cong F^*$ is finite (compact) and $N/V$ is compact, $H/V$ is compact. By Iwasawa's theorem again, we have $H = VK$ is a semidirect product with $K$ compact. This concludes the proof.

We should note that by [**2**, Proposition 4.10] the above hypotheses imply that $G/G_0$ is in [MAP].

COROLLARY 1. *Let $G$ be a compactly generated group in [MAP]. Assume that $G/G_0$ is abelian. Then $G$ contains an open normal subgroup $H$ such that*

(1) *$H = VK$ is a direct product of a vector group $V$ normal in $G$ and a compact group $K$ and*

(2) *$G/H$ is a finitely generated abelian group.*

PROOF. Except for the statement that $V$ is normal in $G$, this is an obvious consequence of the characterization of [SIN] groups [**2**, Theorem 2.13]. The full statement is obtained from Theorem 4 as follows. Let $H_1 = VK_1$ be the subgroup guaranteed by Theorem 4. By Theorem 1, for example, $K_1$ contains an open normal subgroup $K$ of finite index such that $H = VK$ is a direct product. Since $H$ contains $G_0$ and $G/G_0$ is abelian, $H$ is normal in $G$. Clearly $G/H$ is both finitely generated and abelian.

COROLLARY 2. *Let $G$ be a compactly generated group in [MAP]. Let $V$ be the normal vector subgroup of $G$ whose existence is guaranteed by Theorem 2 and assume that $G/V$ is abelian. Then $G = LK$ is a semidirect* *product of a compact (abelian) group $K$ with a normal subgroup $L$ containing $V$ such that $L/V \simeq Z^n$ for some nonnegative integer $n$.*

PROOF. Since $G/V$ is a compactly generated abelian group with no vector component, $G/V=(L/V)(F/V)$ is a direct product with $L/V \simeq Z^n$ for some $n$ and $F/V$ compact. Furthermore by Iwasawa's theorem $F=VK$ is a semidirect product of $V$ with a compact group $K$. We wish to show that $K$ splits in $G$. A projection of $G$ onto $K$ is given by the following composition:

$$G \xrightarrow{\pi_1} G/V = (L/V)(F/V) \xrightarrow{\pi_2} F/V = (VK)/V \xrightarrow{\varphi} K/(K \cap V) = K.$$

Here $\pi_1$ and $\pi_2$ are natural projections and, since $K$ is compact, the natural isomorphism $\varphi$ is also a homeomorphism [3, Theorem 5.33]. Thus if $k \in K$, we have $\varphi\pi_2\pi_1(k)=\varphi\pi_2(kV) = \varphi(kV)=k$. By [4, Theorem 1, p. 8] it follows that $G$ is a semidirect product of $L$ with $K$.

## REFERENCES


1. S. Grosser and M. Moskowitz, *On central topological groups*, Trans. Amer. Math. Soc. **127** (1967), 317–340. MR **35** #292.

2. ———, *Compactness conditions in topological groups*. I, II, J. Reine Angew. Math. **146** (1971), 1–40.

3. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. 1: *Structure of topological groups. Integration theory, group representations*, Die Grundlehren der math. Wissenschaften, Band 115, Academic Press, New York; Springer-Verlag, Berlin, 1963. MR **28** #158.

4. K. H. Hofmann and P. Mostert, *Splitting in topological groups*, Mem. Amer. Math. Soc. No. 43 (1963). MR **27** #1529.

5. H. Leptin and L. Robertson, *Every locally compact map group is unimodular*, Proc. Amer. Math. Soc. **19** (1968), 1079–1082. MR **37** #6397.

6. B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc. (3) **1** (1951), 178–187. MR **13**, 316.

7. T. W. Wilcox, *On the structure of maximally almost periodic groups*, Bull. Amer. Math. Soc. **73** (1967), 732–734. MR **35** #4344.

8. ———, *On the structure of maximally almost periodic groups*, Math. Scand. **23** (1968), 221–232. MR **40** #5788.



DEPARTMENT OF MATHEMATICS, WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT 06457 (Current address of Lewis Robertson)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627

*Current address* (Theodore W. Wilcox): Mathematisches Institut der Technischer Universität, (8) Munchen 2, Arcisstrasse 21, Federal Republic of Germany