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THE ENTIRE FUNCTIONS SEPARATE βC

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ABSTRACT. The entire functions separate the points of the Stone-Čech compactification of the complex plane.

It was asked by N. L. Alling [**1**, p. 95] whether the entire functions separate the points of the Stone-Čech compactification of the complex plane C, and we show here that this is the case. Let βC denote the Stone-Čech compactification of C (i.e. the maximal ideal space of the Banach algebra of all bounded, continuous real valued functions on C). The result is an easy consequence of the following lemma. (All closures are taken with respect to βC.)

LEMMA. *Let $p \in \beta C$ and let U be an open subset of C with a (real) analytic boundary such that $p \in \bar{U}$. Then there exists an open set $V \subset U$ with $p \in \bar{V}$ such that each component of V is bounded and simply-connected.*

PROOF. Consider the two sets

$$U_1 = \bigcup_{k \geqq 0} \{z \in U : 2k - 1 < |z| < 2k + \tfrac{1}{2}\},$$

$$U_2 = \bigcup_{k \geqq 0} \{z \in U : 2k < |z| < 2k + \tfrac{3}{2}\}.$$

Then $U = U_1 \cup U_2$ and the sets U_1, U_2 have bounded components. Now either $p \in \bar{U}_1$ or $p \in \bar{U}_2$, say $p \in \bar{U}_1$.

There exist open subsets V_1 and V_2 of U_1 with simply-connected components such that each component of U_1 is the union of a component of V_1 and a component of V_2. (The existence of such sets is easy, since U_1 is the intersection of two sets with analytic boundaries. We will comment more on this after the proof of the theorem.) The lemma now follows for we have either $p \in \bar{V}_1$ or $p \in \bar{V}_2$.

THEOREM. *The entire functions separate βC.*

PROOF. Let $p_1, p_2 \in \beta C$, $p_1 \neq p_2$. Since the bounded *real* analytic functions on the plane are dense in the space of all bounded continuous

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functions on the plane, there exists a bounded, real analytic function φ such that $\hat{\varphi}(p_1)=0$ and $\hat{\varphi}(p_2)=1$. Here we are using the following notation: for φ a mapping of C into C which is continuous when considered as a map into the Riemann sphere, $\hat{\varphi}$ denotes the unique extension of φ as a map from βC to the Riemann sphere.

Now, by Sard's theorem, for almost all $0<t<1$ the level set $\{z \in C | \varphi(z)=t\}$ contains no critical points of φ. Thus, for some $0<t_1<t_2<1$, the disjoint open sets $U_1=\{z \in C: \varphi(z)<t_1\}$ and $U_2=\{z \in C: \varphi(z)>t_2\}$ have analytic boundaries, and $p_i \in \bar{U}_i$, $i=1, 2$. From the lemma, there exist open sets $V_1 \subset U_1$ and $V_2 \subset U_2$ having simply-connected, bounded components such that $p_i \in \bar{V}_i$, $i=1, 2$. Because V_1 and V_2 are disjoint and have simply-connected bounded components, it follows from Runge's theorem by a standard argument that there exists an entire function $F(z)$ such that for all $z \in V_1$, $|F(z)|<\frac{1}{4}$ and for all $z \in V_2$, $|F(z)-1|<\frac{1}{4}$. For any such F, $\hat{F}(p_1)\neq\hat{F}(p_2)$, as desired.

REMARK. The smoothness assumptions on the boundary of U (in the lemma) and on φ in the proof of the theorem are not necessary. All that one needs to know is that every open connected set in the plane can be written as the union of two open simply-connected sets, and this has been proved by R. Jones [**2**]. In fact, Jones has shown that if U is an open, connected 2-manifold without boundary which is homeomorphic with a proper subset of a compact 2-manifold then U is a union of two simply-connected regions. In particular, the argument given above will show that if Ω is an open Riemann surface with this property, then the points of $\beta\Omega$ are separated by the analytic function on Ω. This answers the question originally posed by Alling.

ADDED IN PROOF. R. Jones has shown [*Riemann surfaces are unions of two discs*, Notices Amer. Math. Soc. **18** (1971), p. 383, Abstract #683-G4] that every Riemann surface is the union of two open sets, each of which is homeomorphic to the open unit disc in the plane. Consequently, the analytic functions separate the points of $\beta\Omega$ for an arbitrary open Riemann surface Ω.

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