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ON WEYL'S THEOREM AND ITS CONVERSE

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ABSTRACT. In this paper we study Weyl's theorem and von Neumann's converse of Weyl's theorem for the classes of all operators of the form $T_f^{-1}T_g$ and of the form $T_gT_f^{-1}$, where T_g and T_f are Toeplitz operators such that T_f is invertible; and we can prove that Weyl's theorem holds for $T_f^{-1}T_g$ and for $T_gT_f^{-1}$.

Let $L(H)$ be the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H, and let $K(H)$ be the closed ideal of compact operators on H. We define the Weyl spectrum $\omega(A)$ to be $\bigcap \sigma(A+K)$, where $\sigma(A)$ denotes the spectrum of A in $L(H)$ and the intersection is taken over all K in $K(H)$. We say that *Weyl's theorem holds for A if $\sigma(A)-\omega(A)=\sigma_{00}(A)$, where $\sigma_{00}(A)$ denotes the set of all isolated eigenvalues of finite multiplicity of A*. It is known that Weyl's theorem holds for any seminormal operator [1] and for any Toeplitz operator [4]. In 1935 J. von Neumann proved the following striking converse of Weyl's classical theorem [7, Satz II]: If A and B are selfadjoint operators in $L(H)$ such that $\omega(A)=\omega(B)$, then there exists a unitary operator U in $L(H)$ such that UAU^*-B is compact. Following S. K. Berberian [2] we say that A, $B\in L(H)$ are *essentially equivalent* if there exists a unitary operator U in $L(H)$ such that UAU^*-B is compact. It is known that if A and B are normal operators such that $\omega(A)=\omega(B)$, then A and B are essentially equivalent (see e.g. [9]).

Let L^2 and L^∞ denote the Lebesgue spaces of square-integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. If $\varphi\in L^\infty$, the Toeplitz operator induced by φ is the operator T_φ on H^2 defined by $T_\varphi f=P(\varphi f)$; here P stands for the orthogonal projection in L^2 with range H^2. In this paper we shall study Weyl's theorem and its converse for the classes of all operators of the form $T_f^{-1}T_g$ and of the form $T_gT_f^{-1}$, where f, $g\in L^\infty$ and T_f is invertible. In particular, we can prove the following results: (Theorem 1) $\sigma(T_f^{-1}T_g)$ is connected; (Theorem 3) Weyl's theorem holds for $T_f^{-1}T_g$; and (Theorem 4) if

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f_i, $g_i \in H^\infty + C$ where C stands for the space of continuous complex-valued functions on the unit circle, and if $\omega(T_{f_1}^{-1}T_{g_1}) = \omega(T_{f_2}^{-1}T_{g_2})$ is a proper subset of the unit circle, then $T_{f_1}^{-1}T_{g_1}$ and $T_{f_2}^{-1}T_{g_2}$ are essentially equivalent.

Recall that an operator A is a Fredholm operator if it has a closed range and both a finite-dimensional kernel and cokernel. The class F of Fredholm operators constitutes a multiplicative open semigroup in $L(H)$. For any $A \in F$, the index $i(A)$ is defined by the formula $i(A) = \dim[\ker A] - \dim[\operatorname{coker} A]$, and it is known that i is a continuous integer-valued function and is invariant under compact perturbations.

Schechter [**8**] has observed that for any operator A,

$$\omega(A) = \{\lambda \mid A - \lambda I \notin F\} \cup \{\lambda \mid A - \lambda I \in F \quad \text{and} \quad i(A - \lambda I) \neq 0\}.$$

We need the following result of [**4**].

LEMMA A. *If $f \in L^\infty$, then T_f is invertible if and only if T_f is Fredholm of index zero.*

THEOREM 1. *$\sigma(T_f^{-1}T_g)$ is connected.*

PROOF. If $\lambda \notin \sigma(T_f^{-1}T_g)$, $T_f^{-1}T_g - \lambda I$ is invertible and this implies that $T_{g-\lambda f}$ is invertible. Hence $\sigma(T_f^{-1}T_g)$ contains $R(g/f)$, the essential range of g/f. We shall generalize a method in [**11**] to prove our theorem and hence we only give an outline of the proof. It suffices to show that if Γ is any simple closed curve in the complex plane which is disjoint from $\sigma(T_f^{-1}T_g)$, then $\sigma(T_f^{-1}T_g)$ lies entirely inside or entirely outside Γ. Since $T_{g-\lambda f}$ and $T_{(g-\lambda f)^{-1}}$ are invertible for each $\lambda \in \Gamma$, the equations $T_{g-\lambda f}h = 1$ and $T_{(g-\lambda f)^{-1}}k = 1$ have solutions $h = h_\lambda$ and $k = k_\lambda$ in H^2 which can be shown [**11**, p. 579] to satisfy two differential equations whose solutions are

$$h_\lambda = h_{\lambda_0} \exp\left(\int_{\lambda_0}^{\lambda} P((g/f) - \mu)^{-1}\, d\mu\right) \tag{1}$$

and

$$k_\lambda = k_{\lambda_0} \exp\left(-\int_{\lambda_0}^{\lambda} P((g/f) - \mu)^{-1}\, d\mu\right) \tag{2}$$

respectively, where λ_0 is a fixed point of Γ. If one takes the path of integration to be the entire curve Γ, then it can be shown [**11**, p. 580] from (1) that $R(g/f)$ lies either entirely inside or entirely outside Γ. In the latter case, say, (1) and (2) show how to continue h_λ and k_λ analytically to the inside of Γ. Now there is an explicit formula which gives the solution of the equation

$$(T_f^{-1}T_g - \lambda I)\varphi = \psi \tag{3}$$

in terms of h_λ and k_λ for $\lambda\notin\sigma(T_f^{-1}T_g)$, in fact

$$\varphi = \varphi_\lambda = h_\lambda P(((g/f) - \lambda)^{-1}k_\lambda\psi) \tag{4}$$

if $\psi\in H^\infty$. But then (4) shows us how to continue $\varphi=\varphi_\lambda$ analytically to the inside of Γ, and this continuation will provide the unique solution of (3) [**11**, pp. 581–582]. Thus we have shown that $\sigma(T_f^{-1}T_g)$ lies entirely outside Γ. This completes the proof of the theorem.

Since $T_f^{-1}T_g$ and $T_gT_f^{-1}$ are similar, we have the following corollary.

COROLLARY 1.1. *$\sigma(T_gT_f^{-1})$ is connected.*

If f is in a special subclass of functions of L^∞, we can have the following theorem, in particular, we generalize some results in [**6**].

THEOREM 2. *If $f=uhk$ where h, $\bar{k}\in(H^\infty)^{-1}$, $u\in H^\infty+C$ such that T_u is invertible, then $\sigma(T_f^{-1}T_g)=\sigma(T_{g/f})=\sigma(T_gT_f^{-1})$.*

PROOF. It suffices to prove $\sigma(T_f^{-1}T_g)=\sigma(T_{g/f})$ only. Since

$$T_f^{-1}T_g = (T_uT_h)^{-1}(T_{g/hk}T_u^{-1})(T_uT_h),$$

we have $\sigma(T_f^{-1}T_g)=\sigma(T_{g/hk}T_u^{-1})$.

By Schechter's characterization of Weyl spectrum and Lemma A, we have $\sigma(T_f^{-1}T_g)=\omega(T_f^{-1}T_g)$ [**6**]. Since $u\in H^\infty+C$ and T_u is invertible, we have $T_u^{-1}=T_{1/u}+K$, where K is a compact operator [**10**, Theorem 6] and $1/u\in H^\infty+C$ [**5**]; and since Weyl's theorem holds for any Toeplitz operator and $\sigma(T_f)=\omega(T_f)$ [**4**], we have

$$\begin{aligned}\sigma(T_f^{-1}T_g) &= \sigma(T_{g/hk}T_u^{-1}) = \omega(T_{g/hk}T_u^{-1})\\ &= \omega(T_{g/f} + K) = \omega(T_{g/f}) = \sigma(T_{g/f}).\end{aligned}$$

The proof is complete.

THEOREM 3. *Weyl's theorem holds for $T_f^{-1}T_g$.*

PROOF. If there were a $\lambda\in\sigma_{00}(T_f^{-1}T_g)$, this implies that $\{\lambda\}=\sigma(T_f^{-1}T_g)$ by Theorem 1. Since $R(g/f)\subset\sigma(T_f^{-1}T_g)$ and it is never empty, $R(g/f)=\{\lambda\}$. It follows that $g=\lambda f$ a.e. and $T_f^{-1}T_g=\lambda I$ and λ is an eigenvalue for $T_f^{-1}T_g$ of infinite multiplicity, which contradicts the fact $\lambda\in\sigma_{00}(T_f^{-1}T_g)$. Therefore $\sigma_{00}(T_f^{-1}T_g)=\varnothing$.

On the other hand, by Schechter's characterization of Weyl spectrum and Lemma A, we have $\sigma(T_f^{-1}T_g)=\omega(T_f^{-1}T_g)$. Therefore Weyl's theorem holds for $T_f^{-1}T_g$. The proof is complete.

COROLLARY 3.1. *Weyl's theorem holds for $T_gT_f^{-1}$.*

COROLLARY 3.2. *If $f=uhk$ where h, $\bar{k}\in(H^\infty)^{-1}$, $u\in H^\infty+C$ such that T_u is invertible, then $\|T_f^{-1}T_g+K\|\geq\|T_{g/f}\|$ for each compact operator K.*

PROOF. By Theorems 2 and 3, we have

$$\sigma(T_{g/f}) = \sigma(T_f^{-1}T_g) = \omega(T_f^{-1}T_g).$$

Therefore,

$$\|T_{g/f}\| = r(T_{g/f}) = r(T_f^{-1}T_g) \leqq r(T_f^{-1}T_g + K) \leqq \|T_f^{-1}T_g + K\|$$

for each compact operator K, where $r(A)$ denotes the spectral radius of A. The proof is complete.

It is easily seen that $T_{f_1}^{-1}T_{g_1}$ and $T_{f_2}^{-1}T_{g_2}$ are not essentially equivalent in general, even if $\omega(T_{f_1}^{-1}T_{g_1})=\omega(T_{f_2}^{-1}T_{g_2})$; in fact we can take $f_1=f_2=1$, $g_1=z$ and $g_2=\bar{z}$.

THEOREM 4. *If f_i, $g_i\in H^\infty+C$ such that T_{f_i} is invertible for each $i=1, 2$ and if $\omega(T_{f_1}^{-1}T_{g_1})=\omega(T_{f_2}^{-1}T_{g_2})$ is a proper subset of the unit circle, then $T_{f_1}^{-1}T_{g_1}$ and $T_{f_2}^{-1}T_{g_2}$ are essentially equivalent. Similarly for $T_{g_i}T_{f_i}^{-1}$, $i=1, 2$.*

PROOF. If f, $g\in H^\infty+C$ and if T_f is invertible, then $T_f^{-1}=T_{1/f}+K$ and $T_hT_g=T_{hg}+J$, where K and J are compact operators [**10**]. It follows that $T_{f_i}^{-1}T_{g_i}=T_{g_i/f_i}+K_i$, $i=1, 2$, where K_i are compact operators and $\omega(T_{g_1/f_1})=\omega(T_{g_2/f_2})=\sigma(T_{g_i/f_i})=\omega(T_{f_i}^{-1}T_{g_i})$. Therefore, $\sigma(T_{g_i/f_i})$ is a proper subset of the unit circle for each $i=1, 2$, and hence $T_{g_i/f_i}=U_i+J_i$, $i=1, 2$, where U_i are unitary operators and J_i are compact operators [**10**]. It follows that $\omega(U_1)=\omega(U_2)=\sigma(T_{g_i/f_i})$. Hence there is a unitary operator U such that $U_1-UU_2U^*$ is compact [**2**]. It is easy to check that $T_{f_1}^{-1}T_{g_1}-U(T_{f_2}^{-1}T_{g_2})U^*$ is compact for this U. Hence $T_{f_1}^{-1}T_{g_1}$ and $T_{f_2}^{-1}T_{g_2}$ are essentially equivalent. The proof is complete.

COROLLARY 4.1. *If $g_i\in H^\infty+C$, $i=1, 2$, and if $\omega(T_{g_1})=\omega(T_{g_2})$ is a proper subset of the unit circle, then T_{g_1} and T_{g_2} are essentially equivalent.*

COROLLARY 4.2. *If $f_i\in H^\infty+C$ such that T_{f_i} is invertible for each $i=1, 2$ and $\omega(T_{f_1}^{-1})=\omega(T_{f_2}^{-1})$ is a proper subset of the unit circle, then $T_{f_1}^{-1}$ and $T_{f_2}^{-1}$ are essentially equivalent.*

THEOREM 5. *If $f_i\in H^\infty+C$ for each $i=1, 2$, then* $\operatorname{Re}\omega(T_{g_1}T_{f_1}^{-1})=\operatorname{Re}\omega(T_{g_2}T_{f_2}^{-1})$ *for some $g_i\in L^\infty$, $i=1, 2$, if and only if* $\operatorname{Re}(T_{g_1}T_{f_1}^{-1})$ *and* $\operatorname{Re}(T_{g_2}T_{f_2}^{-1})$ *are essentially equivalent, where* $\operatorname{Re}(T_{g_i}T_{f_i}^{-1})$ *denotes the real part of $T_{g_i}T_{f_i}$ for each $i=1, 2$. Similarly for* $\operatorname{Im}(T_{g_i}T_{f_i}^{-1})$.

PROOF. It suffices to prove the real part only. Since $f_i\in H^\infty+C$ for $i=1$, 2 we have

$$\omega(T_{g_i}T_{f_i}^{-1}) = \omega(T_{g_i/f_i}) = \sigma(T_{g_i/f_i}).$$

It is known that $\operatorname{Re}\sigma(T_h)=\sigma(\operatorname{Re} T_h)$ for any Toeplitz operator [**3**]. Therefore, $\omega(\operatorname{Re} T_{g_1/f_1})=\omega(\operatorname{Re} T_{g_2/f_2})$ if $\operatorname{Re}\omega(T_{g_1}T_{f_1}^{-1})=\operatorname{Re}\omega(T_{g_2}T_{f_2}^{-1})$ and hence $\operatorname{Re}(T_{g_1/f_1})$ and $\operatorname{Re}(T_{g_2/f_2})$ are essentially equivalent by von Neumann's theorem [2, Theorem 8.1]. Hence $\operatorname{Re}(T_{g_1}T_{f_1}^{-1})$ and $\operatorname{Re}(T_{g_2}T_{f_2}^{-1})$ are essentially equivalent by the relations $T_{g_i}T_{f_i}^{-1}=T_{g_i/f_i}+K_i$, where K_i are compact operators for $i=1, 2$.

Conversely, if $\operatorname{Re}(T_{g_1}T_{f_1}^{-1})$ and $\operatorname{Re}(T_{g_2}T_{f_2}^{-1})$ are essentially equivalent, then $\omega(\operatorname{Re} T_{g_1}T_{f_1}^{-1})=\omega(\operatorname{Re} T_{g_2}T_{f_2}^{-1})$. Therefore, we have

$$\begin{aligned}\operatorname{Re}\omega(T_{g_i}T_{f_i}^{-1}) &= \operatorname{Re}\omega(T_{g_i/f_i}) = \operatorname{Re}\sigma(T_{g_i/f_i})\\ &= \sigma(\operatorname{Re} T_{g_i/f_i}) = \omega(\operatorname{Re} T_{g_i/f_i}) = \omega(\operatorname{Re} T_{g_i}T_{f_i}^{-1}),\end{aligned}$$

and hence, $\operatorname{Re}\omega(T_{g_1}T_{f_1}^{-1})=\operatorname{Re}\omega(T_{g_2}T_{f_2}^{-1})$. The proof is complete.

COROLLARY 5.1. $\operatorname{Re}\omega(T_{g_1})=\operatorname{Re}\omega(T_{g_2})$ *if and only if* $\operatorname{Re}(T_{g_1})$ *and* $\operatorname{Re}(T_{g_2})$ *are essentially equivalent.*

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