PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 33, Number 2, June 1972

ON THE ARITHMETIC MEAN OF FOURIER-STIELTJES COEFFICIENTS[1]

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ABSTRACT. Let $\{a_n\}_{n=0}^{\infty}$ be the cosine Fourier-Stieltjes coefficients of the Borel measure μ and $\{a_0, (a_1+\cdots+a_n)/n\}_{n=1}^{\infty}=\{(Ta)_n\}_{n=0}^{\infty}$ be the sequence of their arithmetic means. Then $\sum_{n=0}^{\infty}(Ta)_n \cos nx$ is a Fourier-Stieltjes series. Moreover, (a) $\sum_{n=0}^{\infty}(Ta)_n \cos nx$ is a Fourier series if and only if $(Ta)_n \to 0$ at infinity or, equivalently, the measure μ is continuous at the origin, (b) $\sum_{n=1}^{\infty}(Ta)_n \sin nx$ is a Fourier series if and only if the function $x^{-1}\mu([0, x))$ is in $L^1[0, \pi]$. These results form the best possible analogue of a theorem of G. Goes, concerning arithmetic means of Fourier-Stieltjes sine coefficients, and improve considerably the theorems of L. Fejér and N. Wiener on the inversion and quadratic variation of Fourier-Stieltjes coefficients.

1. **Introduction.** Let $M[0, \pi]$ be the set of bounded regular Bore measures on $[0, \pi]$, and let

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos nt \, d\mu(t), \quad b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt \, d\mu(t) \qquad (n = 0, 1, 2, 3, \cdots)$$

be the Fourier-Stieltjes cosine and sine coefficients of the measure μ in $M[0, \pi]$. The sequences of their arithmetic means $\{(Ta)_n\}_{n=0}^{\infty}$ and $\{(Tb)_n\}_{n=1}^{\infty}$ are defined as follows

$$(Ta)_0 = a_0, \quad (Ta)_n = \frac{1}{n}(a_1 + \cdots + a_n), \quad (Tb)_n = \frac{1}{n}(b_1 + \cdots + b_n) \qquad (n = 1, 2, 3, \cdots).$$

G. H. Hardy first studied in [6] the transformation T. He showed that if $\sum_{n=0}^{\infty} a_n \cos nx$ is the Fourier series of a function in $L^p[0, \pi]$, then so is the series $\sum_{n=0}^{\infty}(Ta)_n \cos nx$ for $1 \leqq p < \infty$, and the same is true for sine series.

Presented to the Society, January 18, 1972; received by the editors June 28, 1971.

AMS 1969 *subject classifications*. Primary 4252, 4210, 4220.

Key words and phrases. Fourier-Stieltjes coefficients, Borel measure, discrete measure, inversion, arithmetic mean, Fourier coefficients, conjugate function, conjugate series.

[1] Part of the author's Ph.D. dissertation written at the Illinois Institute of Technology.

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M. Kinukawa and S. Igari in [9] proved that, if $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier series, then the conjugate series $\sum_{n=1}^{\infty} (Tb)_n \cos nx$ is a Fourier series also. In [4], G. Goes using these results proved the following:

THEOREM A. *If $\sum_{n=1}^{\infty} b_n \sin nx$ is a Fourier-Stieltjes series, then both $\sum_{n=1}^{\infty} (Tb)_n \sin nx$ and its conjugate $\sum_{n=1}^{\infty} (Tb)_n \cos nx$ are Fourier series.*

Our purpose here is to study the properties of the transformation T of arithmetic means on Fourier-Stieltjes cosine coefficients. We obtain the best possible analogue of the theorem of G. Goes. We prove that if $\sum_{n=0}^{\infty} a_n \cos nx$ is a Fourier-Stieltjes series, then $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ is also a Fourier-Stieltjes series. Moreover,

(a) $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ is a Fourier series if and only if $(Ta)_n \to 0$ at infinity or, equivalently, the measure is continuous at the origin,

(b) the conjugate series $\sum_{n=1}^{\infty} (Ta)_n \sin nx$ is a Fourier series if and only if the function $x^{-1}\mu([0, x))$ is in $L^1[0, \pi]$.

The approach in [4] is not applicable for the case of Fourier-Stieltjes cosine coefficients, since T does not map sequences of Fourier cosine coefficients into sequences of Fourier sine coefficients [10]. The technique used here is somewhat similar to that of G. H. Hardy [6], [7], R. R. Goldberg [5], and A. Zygmund [12]. This method also yields Theorem A and Corollaries 3.2 and 3.4. The latter are a considerable improvement of the formula of L. Fejér [2] on recovering the discrete component of a measure from its Fourier-Stieltjes coefficients, and the theorem of N. Wiener [11] on quadratic variation of Fourier-Stieltjes coefficients (see [13, pp. 107–108]).

2. **Preliminary lemmas.** At first we study a certain mapping: $\mu \to S\mu$ of $M[0, \pi]$ into $L^1[0, \pi]$, and we examine closely the conjugate (the periodic Hilbert transform) of the even function $S\mu(|x|)$. The properties of the transformation T depend on those of the mapping S. Throughout this paper $S\mu$, unless stated to the contrary, represents the function defined below.

LEMMA 2.1. *Let $K(x, t) = t^{-1}$ for $0 < x \leqq t < \pi$, zero elsewhere in $[0, \pi] \times [0, \pi]$, and define*

$$S\mu(x) = \int_0^{\pi} K(x, t)\, d\mu(t), \quad \textit{for } \mu \in M[0, \pi], \qquad 0 \leqq x \leqq \pi.$$

Note that this makes $S\mu(0) = 0$. Then

(a) *$S\mu \in L^1[0, \pi]$, and for $0 < y \leqq \pi$,*

$$\frac{1}{y} \cdot \mu((0, y)) = -S\mu(y) + \frac{1}{y} \int_0^y S\mu(x)\, dx,$$

(b) $S\mu \in L^p[0, \pi]$, *if there exists* $0<\delta<\pi$ *such that*

$$\int_{(0,\delta)} t^{-1/q} d\,|\mu|\,(t) < \infty, \qquad p + q = pq.$$

PROOF. Part (b) follows simply from Minkowski's inequality for iterated integrals. To prove (a), let $0<y\leqq\pi$ and assume without loss of generality $\mu\in M[0, \pi]$ is nonnegative. Then using Fubini's theorem we have

$$\begin{aligned}\int_0^\pi S\mu(x)\,dx &= \int_0^y dx \int_{[0,y)} K(x, t)\,d\mu(t) + \int_0^y dx \int_y^\pi K(x, t)\,d\mu(t)\\ &= \int_{(0,y)} d\mu(t) \int_{(0,y]} K(x, t)\,dx\,d\mu(t) + \int_y^\pi d\mu(t) \int_0^y K(x, t)\,dx\\ &= \mu((0, y)) + yS\mu(y).\end{aligned}$$

LEMMA 2.2. *Let* $\mu\in[0, \pi]$, *and let* $f(x)$ *be the periodic extension of* $S\mu(|x|)$. *Then its conjugate function* $\tilde f$ *is in* $L^1[-\pi, \pi]$ *if and only if the function* $x^{-1}\mu((0, x))$ *is in* $L^1[0, \pi]$.

PROOF. Clearly $f\in L^1[-\pi, \pi]$ by Lemma 2.1(a). Its conjugate $\tilde f$ exists a.e., is an odd function, and on $[0, \pi]$ it is given by

$$\begin{aligned}\pi\tilde f(x) &= \left(\int_{x/2}^\pi + \int_0^{x/2}\right)\left(\frac{1}{2}\cot\frac{x+t}{2} + \frac{1}{2}\cot\frac{x-t}{2}\right) S\mu(t)\,dt\\ &= f_1(x) + f_2(x).\end{aligned}$$

In order to prove the theorem, it suffices to show that f_1 is an integrable function, and that f_2 is integrable if and only if $x^{-1}\mu((0, x))$ is in $L^1[0, \pi]$. For $0<x\leqq\pi$, put

$$\phi(x) = S(S\mu)(x), \qquad \psi(x) = \frac{1}{x}\int_0^x S\mu(t)\,dt.$$

From Lemma 2.1(a) it follows that ϕ is an integrable function, and that ψ is integrable if and only if $x^{-1}\mu((0, x))$ is integrable on $[0, \pi]$. Thus, in proving the theorem, it suffices to show that f_2 is integrable if and only if ψ is integrable.

In order to prove that f_1 is integrable, we may assume that μ is nonnegative. Then $S\mu$ is nonincreasing on $[0, \pi]$ and bounded except possibly near the origin. We consider first f_1, for $2\pi/3\leqq x\leqq\pi$. Put

$$\begin{aligned}f_1(x) &= \int_{x/2}^\pi \frac{1}{2}\cot\frac{x+t}{2} S\mu(t)\,dt + \int_{x/2}^{2x-\pi} \frac{1}{2}\cot\frac{x-t}{2} S\mu(t)\,dt\\ &\quad - \int_{2x-\pi}^\pi \frac{1}{2}\cot\frac{t-x}{2} S\mu(t)\,dt.\end{aligned}$$

Clearly, if $2\pi/3 \leqq x \leqq \pi$ and $x/2 \leqq t \leqq \pi$, then $\pi/2 \leqq (x+t)/2 \leqq \pi$, $\cot(x+t)/2 \leqq 0$, and $0 \leqq S\mu(t) \leqq S\mu(\pi/3)$. Therefore, the first term in this sum is majorized by

$$-S\mu\left(\frac{\pi}{3}\right)\int_{x/2}^{\pi} \cot\frac{x+t}{2}\,dt = S\mu\left(\frac{\pi}{3}\right)\log\left|\sin\frac{3x}{4}\sec\frac{x}{2}\right|.$$

Likewise, the middle term is majorized by $\log|\cos(x/2)\csc(x/4)|$ and, hence, both are integrable on $[2\pi/3, \pi]$. Let $I(x)$ denote the last term. Then using Fubini's theorem and noting that $\pi-2t \geqq 2\pi/3+t$ for $0 \leqq t \leqq \pi/9$, and that $\pi-2t \leqq 2\pi/3+t$ for $\pi/9 \leqq t \leqq \pi/3$, we have

$$\begin{aligned}
\int_{2\pi/3}^{\pi} |I(x)|\,dx &= \frac{1}{2}\int_{2\pi/3}^{\pi} dx\left(\int_{x}^{\pi} - \int_{x}^{2x-\pi}\right)\cot\frac{x-t}{2}\,S\mu(t)\,dt \\
&\leqq \int_{2\pi/3}^{\pi} dx \int_{0}^{\pi-x} \frac{1}{t}\,(S\mu(x-t) - S\mu(x+t))\,dt \\
&= \int_{0}^{\pi/3} \frac{dt}{t}\int_{0}^{\pi-t} (S\mu(x-t) - S\mu(x+t))\,dx \\
&= \int_{0}^{\pi/9} \frac{dt}{t}\left(\int_{(2\pi/3)-t}^{\pi-2t} - \int_{(2\pi/3)+t}^{\pi}\right) S\mu(x)\,dx \\
&\quad + \int_{\pi/9}^{\pi/3} \frac{dt}{t}\left(\int_{(2\pi/3)-t}^{\pi-2t} - \int_{(2\pi/3)+t}^{\pi}\right) S\mu(x)\,dx \\
&= \int_{0}^{\pi/9} \frac{dt}{t}\left(\int_{(2\pi/3)-t}^{(2\pi/3)+t} - \int_{\pi-t}^{\pi}\right) S\mu(x)\,dx \\
&\quad + \int_{\pi/9}^{\pi/3} \frac{dt}{t}\left(\int_{(2\pi/3)-t}^{(2\pi/3)+t} - \int_{\pi-t}^{\pi}\right) S\mu(x)\,dx \\
&< 2\int_{0}^{\pi} S\mu(t)\,dt.
\end{aligned}$$

Now we consider f_1, for $0 \leqq x \leqq 2\pi/3$. Then

$$\begin{aligned}
f_1(x) &= \int_{3x/2}^{\pi}\left\{\frac{1}{2}\cot\frac{x+t}{2} + \frac{1}{2}\cot\frac{x-t}{2}\right\}S\mu(t)\,dt \\
&\quad + \int_{x/2}^{3x/2} \frac{1}{2}\cot\frac{x+t}{2}\,S\mu(t)\,dt - \int_{x/2}^{3x/2} \frac{1}{2}\cot\frac{t-x}{2}\,S\mu(t)\,dt.
\end{aligned}$$

The first and second term in this sum are integrable since they are dominated by $4\phi(3x/2)$ and $xS\mu(x/2)\cot(3x/4)$, respectively, for $0 \leqq x \leqq 2\pi/3$. The last term is also integrable by an argument similar to that given for

$l(x)$. In fact, its integral does not exceed the following

$$\int_{\pi/6}^{\pi/3} \frac{dt}{t}\left\{\int_t^{3t} S\mu(x)\,dx - \int_{(2\pi/3)-t}^{(2\pi/3)+t} S\mu(x)\,dx\right\} + \int_0^{\pi/3} \frac{dt}{t}\left\{\int_t^{3t} S\mu(x)\,dx - \int_{(2\pi/3)-t}^{(2\pi/3)+t} S\mu(x)\,dx\right\} \leqq 2\int_0^{\pi} S\mu(t)\,dt.$$

This completes the proof that f_1 is in $L^1[0, \pi]$.

Finally, we consider $2\psi - f_2$ for $0 \leqq x \leqq \pi$. Since μ depends linearly on μ, we may as well assume $\mu \geqq 0$. Then using the mean value theorem we obtain that $2\psi(x) - f_2(x)$ does not exceed the following:

$$\begin{aligned} &\frac{2}{x}\int_{x/2}^{x} S\mu(t)\,dt + \left|\cot\frac{x}{2} - \frac{2}{x}\right| \int_0^{x/2} S\mu(t)\,dt \\ &\quad + \frac{1}{2}\int_0^{x/2}\left\{\left|\cot\frac{x-t}{2} - \cot\frac{x}{2}\right| + \left|\cot\frac{x+t}{2} - \cot\frac{x}{2}\right|\right\} S\mu(t)\,dt \\ &\leqq \frac{2}{x}\int_{x/2}^{x} S\mu(t)\,dt + \left|\cot\frac{x}{2} - \frac{2}{x}\right| \int_0^{x/2} S\mu(t)\,dt \\ &\quad + \frac{1}{4}\int_0^{x/2}\left(\csc^2\frac{x}{4} + \csc^2\frac{x}{2}\right) tS\mu(t)\,dt \\ &= \omega(x) + \sigma(x) + \theta(x). \end{aligned}$$

Now ω is integrable on $[0, \pi]$, since $S\mu$ is nonincreasing and $\omega(x) \leqq S\mu(x/2)$. On the other hand, the function σ is integrable, since $\cot(x/2) - 2/x$ is bounded for $0 \leqq x \leqq \pi$. Finally, θ is integrable on $[0, \pi]$, because

$$\begin{aligned} \int_0^{\pi} \theta(x)\,dx &\leqq 10\pi^2 \int_0^{\pi} tS\mu(t)\,dt \int_{2t}^{\pi} \frac{dx}{x^2} \\ &= 5\pi^2 \int_0^{\pi} S\mu(t)\,dt - 10\pi \int_0^{\pi} tS\mu(t)\,dt < \infty. \end{aligned}$$

This shows that f_2 is integrable if and only if ψ is integrable, and thus completes the proof of the lemma.

3. **The main results.**

THEOREM 3.1. *Let $\sum_{n=0}^{\infty} a_n \cos nx$ be the Fourier-Stieltjes series of a measure μ in $M[0, \pi]$. Then $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ is also a Fourier-Stieltjes series. Moreover,*

(a) $(Ta)_n = (2/\pi) \int_0^{\pi} S\mu(y) \cos ny\, dy + a_n^* + (2/\pi)\mu(\{0\})$ $(n \geqq 0)$, *where $\{a_n^*\}_{n=0}^{\infty}$ are the Fourier cosine coefficients of a function in $L^p[0, \pi]$ for all $1 \leqq p < \infty$.*

(b) $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ *is a Fourier series if and only if $\mu(\{0\}) = 0$, or equivalently, $(Ta)_n \to 0$ at infinity.*

PROOF. Clearly, (b) follows from (a), Lemma 1.1(a), and the Riemann-Lebesgue theorem. For $n=0$, (a) is simply a restatement for $y=\pi$ of Lemma 1.1(a). In proving (a) for $n\geqq 1$ we may assume μ to be nonnegative. Let

$$\phi_n(t) = -\frac{\sin nt}{t} + \sum_{k=1}^{n} \cos kt, \quad \text{for } 0 < t \leq \pi,\ \phi_n(0) = 0,$$

$$a_n^* = \frac{1}{n\pi}\int_0^\pi \phi(t)d\mu(t).$$

Then using Fubini's theorem we have

$$\begin{aligned}\frac{\pi}{2}(Ta)_n - \frac{\pi}{2}a_n^* &= \frac{1}{n}\int_0^\pi d\mu(t)\int_0^n \cos tx\, dx\\ &= \mu(\{0\}) + \int_0^\pi d\mu(t)\int_0^\pi K(y, t)\cos ny\, dy\\ &= \mu(\{0\}) + \int_0^\pi dy\int_0^\pi K(y, t)\, d\mu(t)\cos ny\\ &= \mu(\{0\}) + \int_0 S\mu(y)\cos ny\, dy.\end{aligned}$$

Finally, since $\{\phi_n(t)\}_{n=1}^\infty$ are uniformly bounded on $[0, \pi]$, $\sum_{n=1}^\infty |a_n^*|^q < \infty$ for all $1<q\leqq 2$ [**13**, p. 55]. This, together with the Hausdorff-Young theorem, implies that $\{a_n\}_{n=1}^\infty$ are the Fourier cosine coefficients of a function in $L^p[0, \pi]$ for all $1\leqq p<\infty$.

COROLLARY 3.2. *Let $\{c_n\}_{n=-\infty}^\infty$ be the (exponential) Fourier-Stieltjes coefficients of a measure μ in $M(G)$—the set of bounded regular Borel measures on the circle group $G=(-\pi, \pi]$. Then*

$$\left\{\frac{1}{n}\sum_{k=-n}^{n} c_k e^{ikp} - \frac{1}{\pi}\mu(\{p\})\right\}_{n=1}^\infty, \qquad \left\{\frac{1}{2n+1}\sum_{k=-n}^{n} |c_k|^2 - \frac{1}{4\pi^2}\sum_{x\in G} |\mu(\{x\})|^2\right\}_{n=1}^\infty$$

are both sequences of Fourier cosine coefficients, for each p in G.

PROOF. Since the second sequence above can be obtained from the first and the first sequence from its special case $p=0$, it suffices to prove only the first part of the statement for $p=0$. Let $\lambda(E)=\frac{1}{2}(\mu(E)+\mu(-E))$. Then $\{c_k+c_{-k}+\pi^{-1}\mu(\{0\})\}_{k=0}^\infty$ are the Fourier-Stieltjes cosine coefficients of the measure λ. To complete the proof we apply Theorem 3.1(a) to the last sequence, and observe that $\{-c_0/n\}_{n=1}^\infty$ are Fourier cosine coefficients by the Riesz-Fischer theorem.

THEOREM 3.3. *Let $\sum_{n=0}^\infty a_n \cos nx$ be the Fourier-Stieltjes series of a measure μ in $M[0, \pi]$. Then the series $\sum_{n=0}^\infty (Ta)_n \sin nx$ is a Fourier series if and only if the function $x^{-1}\mu([0, x))$ is in $L^1[0, \pi]$.*

PROOF. Let $f(x)=S\mu(|x|)$ for $-\pi<x\leqq\pi$ and 2π periodic elsewhere. Suppose $x^{-1}\mu([0,x))$ is in $L^1[0,\pi]$. Then $\mu(\{0\})=0$ and $\tilde{f}\in L^1[0,\pi]$ by Lemma 2.2. According to Theorem 3.1(b), the series $\sum_{n=0}^{\infty}(Ta)_n\cos nx$ is the Fourier series of an even function $f+h$, where h and $\tilde{h}$ belong in $L^p[-\pi,\pi]$ $(1\leqq p<\infty)$ by the theorem of M. Riesz. Since $\tilde{f}\in L^1[-\pi,\pi]$, it follows that the conjugate series $\sum_{n=1}^{\infty}(Ta)_n\sin nx$ is the Fourier series of $\tilde{f}+\tilde{h}$.

Now suppose $\sum_{n=1}^{\infty}(Ta)_n\sin nx$ is the Fourier series of an odd function g in $L^1[-\pi,\pi]$. Then $(Ta)_n\to 0$ at infinity by the Riemann-Lebesgue theorem. From Theorem 3.1(a) it follows that $\sum_{n=0}^{\infty}(Ta)_n\cos nx$ is the Fourier series of an even function $f+h$, where $h\in L^p[-\pi,\pi]$ and $f(x)=S\mu(|x|)$. Clearly, the Fourier series $\sum_{n=1}^{\infty}(Ta)_n\sin nx$ is Abel summable a.e. to $g(x)$. On the other hand, the series $\sum_{n=1}^{\infty}(Ta)_n\sin nx$, being the conjugate series of $\sum_{n=0}^{\infty}(Ta)_n\cos nx$, is Abel summable to the conjugate function $\tilde{f}+\tilde{h}$. Hence, $g(x)=\tilde{f}(x)+\tilde{h}(x)$ a.e. This implies that $\tilde{f}\in L^1[-\pi,\pi]$, since $\tilde{h}\in L^p[-\pi,\pi]$. Hence, the functions $x^{-1}\mu([0,x))$ is in $L^1[0,\pi]$ by Lemma 2.2.

COROLLARY 3.4. *Let $\{c_n\}_{n=-\infty}^{\infty}$ be the Fourier-Stieltjes coefficients of a measure μ in $M(G)$. Then the sequence*

$$\left\{\frac{1}{n}\sum_{k=-n}^{n}c_k-\frac{1}{\pi}\mu(\{0\})\right\}_{n=1}^{\infty}$$

is a sequence of Fourier sine coefficients if and only if the function $x^{-1}\mu((-x,x))$ is in $L^1[0,\pi]$.

Finally, we can combine Theorem A, Theorem 3.1, and Theorem 3.3 into one theorem concerning arithmetic means of Fourier-Stieltjes coefficients of measures in $M(G)$. Let λ be the measure defined in the proof of Corollary 3.2. By applying Theorems 3.1 and 3.3 to the Fourier-Stieltjes series for the measure λ, Theorem A to the Fourier-Stieltjes series for the measure $\mu-\lambda$, and then adding, we obtain the following:

THEOREM 3.4. *Let $\sum_{n=-\infty}^{\infty}c_ne^{inx}$ be the Fourier-Stieltjes series of a measure μ in $M(G)$, and let $(Tc)_0=c_0$, $(Tc)_n=(1/|n|)\sum_{0<k/n\leq 1}c_k$ for $|n|\geqq 1$. Then $\sum_{n=-\infty}^{\infty}(Tc)_ne^{inx}$ is also a Fourier-Stieltjes series. Moreover,*

(a) *$\sum_{n=-\infty}^{\infty}(Tc)_ne^{inx}$ is a Fourier series if and only if $(Tc)_n\to 0$ at infinity, or, equivalently, the measure μ is continuous at the origin;*

(b) *the conjugate series $-i\sum_{n=-\infty}^{\infty}\operatorname{sign}(n)(Tc)_ne^{inx}$ is a Fourier series if and only if the function $x^{-1}\mu((-x,x))$ is in $L^1(G)$.*

REMARKS. (a) The analogue of Theorem 3.3(a) was proved for probability measures on the real line by M. Girault [3], and by R. P. Boas, Jr. and S. Izumi [1] for absolutely convergent Fourier series. (b) If a measure μ satisfies the condition of Lemma 1.1(b), and $\sum_{n=0}^{\infty} a_n \cos nx$ is its Fourier-Stieltjes series, then $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ is the Fourier series of a function in $L^p[0, \pi]$ by Theorem 3.1(a); this result was proved by M. and S. Izumi [**8**] in the case of absolutely continuous measure μ. (c) A necessary condition for $\sum_{n=1}^{\infty} (Ta)_n \sin nx$ to be a Fourier series is that $\sum_{n=1}^{\infty} (|a_1+\cdots+a_n|/n^2)<\infty$ [**13**, p. 286]. (d) Theorem A and Theorem 3.3(b) are also true for the real line—the proofs are similar. (e) Theorem 3.4(a) remains true if we set $\psi(k/n)c_k$ in the definition of $(Tc)_n$ in place of c_k, and ψ is a function of bounded variation with support in [0, 1].

The author wishes to thank Professor Günther Goes for helpful advice and criticism during the preparation of this research, and the referee, whose comments improved the exposition in the article.

ADDENDUM. Professor G. Goes brought to my attention that the fact in Theorem 3.1, that $\sum_{n=0}^{\infty} (Ta)_n \cos nx$ is a Fourier-Stieltjes series, was already observed in [**4**].

REFERENCES

1. R. P. Boas, Jr. and S. Izumi, *Absolute convergence of some trigonometric series.* I, J. Indian Math. Soc. **24** (1960), 191–210. MR **23** #A1198.

2. L. Fejér, *Über die Bestimung des sprunges einer Funktion aus ihrer Fourierreihe*, J. Reine Angew. Math. **142** (1913) 165–168.

3. M. Girault, *Transformation de fonctions caractéristiques par intégration*, C. R. Acad. Sci. Paris **238** (1954), 2223–2224. MR **16**, 52.

4. G. Goes, *Arithmetic means of Fourier-Stieltjes-sine-coefficients*, Proc. Amer. Math. Soc. **14** (1963), 10–11. MR **26** #2791.

5. R. R. Goldberg, *Averages of Fourier coefficients*, Pacific J. Math. **9** (1959), 695–699. MR **22** #2842.

6. G. H. Hardy, *Notes on some points in the integral calculus*, Messenger Math. **58** (1929), 50–52.

7. ———, *On a theorem of Paley and Wiener*, Proc. Cambridge Philos. Soc. **33** (1937), 1–5.

8. M. Izumi and S. Izumi, *On a Hardy's theorem*, Proc. Japan Acad. **44** (1968), 418–423. MR **38** #2519.

9. M. Kinukawa and S. Igari, *Transformations of conjugate functions*, Tôhoku Math. J. (2) **13** (1961), 274–280. MR **26** #2829.

10. C. T. Loo, *Note on the properties of Fourier coefficients*, Amer. J. Math. **71** (1949), 269–282. MR **10**, 603.

11. N. Wiener, *The quadratic variation of a function and its Fourier constants*, Mass. J. Math. **3** (1924), 73–94.

12. A. Zygmund, *Some points in the theory of trigonometric and power series*, Trans. Amer. Math. Soc. **36** (1934), 586–617, p. 615.

13. ———, *Trigonometrical series.* Vol. I, 2nd ed., Cambridge Univ. Press, New York, 1959. MR **21** #6498.

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