ON THE RING OF QUOTIENTS OF A GROUP RING

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Abstract. In this paper the following extension of a result of Martha Smith is proved. Theorem. Let \( K[G] \) be a group ring which is an order in a ring \( Q \). Then the center of \( K[G] \) is an order in the center of \( Q \).

Let \( K[G] \) denote the group ring of \( G \) over the field \( K \). The main result of this paper is

Theorem. Let \( K[G] \) be a group ring which is an order in a ring \( Q \). Then the center of \( K[G] \) is an order in the center of \( Q \).

This was first proved by Martha Smith [2] in a number of special cases and later proved [1] for all semiprime group rings. We follow the notation of [1]. Thus, in particular, \( \Delta(G) \) denotes the finite conjugate subgroup of \( G \) and \( \theta : K[G] \to K[\Delta(G)] \) is the natural projection.

Lemma 1. Let \( H \subseteq \Delta(G) \) be a finitely generated normal subgroup of \( G \). Then

(i) \( [G:C_G(H)] < \infty \).

(ii) \( H \) has a torsion free central subgroup \( Z \) of finite index which is normal in \( G \).

(iii) The ring of quotients \( K[Z]^{-1}K[H] \) is \( K \)-isomorphic to \( F^1[H/Z] \), a twisted group ring of the finite group \( H/Z \) over the field \( F = K[Z]^{-1}K[Z] \).

Proof. (i) Let \( H = \langle h_1, h_2, \ldots, h_n \rangle \). Since \( H \subseteq \Delta(G) \) we have \( [G:C_G(h_i)] < \infty \). Hence \( C_G(H) = \cap_i C_G(h_i) \) has finite index in \( G \).

(ii) By (i) we see that \( Z(H) = H \cap C_G(H) \) has finite index in \( H \) and certainly \( Z(H) \) is normal in \( G \). Since \( H \) is finitely generated and \( [H:Z(H)] < \infty \), it follows that \( Z(H) \) is a finitely generated abelian group. Thus \( Z(H) = T \times A \) where \( T \) is a finitely generated torsion free abelian group and \( A \) is finite of order \( k \). If \( Z = \{ x^k \mid x \in Z(H) \} \), then clearly \( Z \) is normal in \( G \) and \( Z \) is a torsion free central subgroup of \( H \) of finite index.

(iii) Now by Lemma 2.4 of [1] no nonzero element of \( K[Z] \) is a zero divisor in \( K[G] \) since \( Z \) is torsion free abelian. Since \( Z \) is central in \( K[H] \),
it is then trivial to form the ring of quotients \( E = \mathbb{K}[Z]^{-1}\mathbb{K}[H] \). This is the set of all formal fractions \( \eta^{-1}x \) with \( \eta \in \mathbb{K}[Z] \), \( \eta \neq 0 \), \( x \in \mathbb{K}[H] \) and with the usual identifications made. If \( F = \mathbb{K}[Z]^{-1}\mathbb{K}[Z] \), then \( F \) is certainly a central subfield of \( E \) so \( E \) is an \( F \)-algebra. For each \( x \in H/Z \) let \( \bar{x} \in H \) be a coset representative. Then it is easy to see that \( \{ \bar{x} | x \in H/Z \} \) is an \( F \)-basis for the associative algebra \( E \). Moreover for \( x, y \in H/Z \) we have

\[
\bar{x} \bar{y} = z \bar{x} \bar{y}
\]

for some \( z \in \mathbb{Z} \).

Since \( z \) is a nonzero element of \( F \) we conclude that \( E \cong F^4[H/Z] \).

**Lemma 2.** Let \( \alpha \) be an element of \( \mathbb{Z}(\mathbb{K}[G]) \), the center of the group ring. Then \( \alpha \) is a zero divisor in \( \mathbb{K}[G] \) if and only if it is a zero divisor in \( \mathbb{Z}(\mathbb{K}[G]) \).

**Proof.** If \( \alpha \) is a zero divisor in \( \mathbb{Z}(\mathbb{K}[G]) \) then it is certainly a zero divisor in \( \mathbb{K}[G] \). Assume now that \( \alpha \) is a zero divisor in \( \mathbb{K}[G] \) and let \( \gamma \in \mathbb{K}[G], \gamma \neq 0 \) with \( \alpha \gamma = 0 \). Let \( H = (\text{Supp } \alpha) \). Since \( \alpha \) is central, \( H \) is a finitely generated subgroup of \( \Delta(G) \) which is normal in \( G \). We use that notation and results of Lemma 1. Then \( \alpha \in E = \mathbb{K}[Z]^{-1}\mathbb{K}[H] \) which is a finite dimensional algebra over \( F \) and hence \( \alpha \) satisfies a polynomial over \( F \). By rationalizing the denominators we can assume that \( \alpha^n \beta = 0 \) where \( \beta = \beta_0 + \beta_1 \alpha + \cdots + \beta_r \alpha^r \) with \( \beta_i \in \mathbb{K}[Z], \beta_0 \neq 0 \).

Now the finite group \( G/C_G(H) \) acts on \( \mathbb{K}[H] \) and let \( x_1 = 1, x_2, \cdots, x_r \) be a full set of coset representatives for \( C_G(H) \) in \( G \). Set \( \tilde{\beta} = \beta_1 \alpha^1 + \cdots + \beta_r \alpha^r \). Since \( Z \) is normal in \( G \) and \( \alpha \) is central in \( \mathbb{K}[G] \) we see that all \( \beta_i \alpha^i \) are polynomials in \( \alpha \) with coefficients in \( \mathbb{K}[Z] \). Since \( Z \) is abelian this then implies that all \( \beta_i \alpha^i \) commute and hence clearly \( \tilde{\beta} \) is central in \( \mathbb{K}[G] \). Now

\[
\tilde{\beta} = (\beta_0^1 \beta_1^2 \cdots \beta_r^r) + \delta \alpha
\]

for a suitable \( \delta \in \mathbb{K}[G] \) so we have

\[
\tilde{\beta} \gamma = \beta_0^1 \beta_1^2 \cdots \beta_r^r \gamma.
\]

Thus by Lemma 2.4 of [1], \( \tilde{\beta} \gamma = 0 \) since \( \beta_0 \neq 0 \) implies that \( \beta_0 \alpha^i \) is not a zero divisor in \( \mathbb{K}[G] \). Hence \( \tilde{\beta} \neq 0 \).

Now \( \alpha^m \tilde{\beta} = \alpha \beta = 0 \). Since \( \alpha^m \beta \neq 0 \) we can choose \( m \geq 0 \) maximal with \( \alpha^m \beta \neq 0 \). Then \( \alpha(\alpha^n \beta) = \alpha^{m+1} \beta = 0 \). Finally \( \alpha^n \beta \in \mathbb{Z}(\mathbb{K}[G]) \) so \( \alpha \) is a zero divisor in \( \mathbb{Z}(\mathbb{K}[G]) \) and the result follows.

A ring \( R \) is said to be self-injective if \( R \) as a right \( R \)-module is injective. The following lemma is well known.

**Lemma 3.** Let \( R \) be a self-injective ring. Then every proper finitely generated left ideal has a proper right annihilator.
Proof. Let $I$ be the left ideal $I = R x_1 + R x_2 + \cdots + R x_n$ and assume that $I$ has no proper right annihilator. Define
\[ \sigma: R \to R + R + \cdots + R \quad (n \text{ times}) \]
by
\[ \sigma(r) = (\alpha_1 r, \alpha_2 r, \cdots, \alpha_n r) \]
for $r \in R$. This is clearly an $R$-homomorphism. If $\sigma(r) = 0$ then $Ir = 0$ so $r = 0$. This $\sigma$ is an injection.

Since $R$ is self-injective, there exists a back map $\tau$. Say
\[ \tau(0, 0, \cdots, 1, \cdots, 0) = \beta_i \in R \]
where the 1 is in the $i$th position. Then
\[ 1 = \tau \sigma(1) = \tau(\alpha_1, \alpha_2, \cdots, \alpha_n) = \beta_1 \alpha_1 + \beta_2 \alpha_2 + \cdots + \beta_n \alpha_n \in I \]
and $I = R$.

The following is known for ordinary group rings.

Lemma 4. Let $F'[W]$ be a twisted group ring of the finite group $W$ over the field $F$. Then $F'[W]$ is self-injective.

Proof. Since $I$ is a scalar multiple of $1 \in F'[W]$ we may assume without loss of generality that $I = 1$. Let $\tau: F'[W] \to F$ be the trace map. That is, if $a = \sum a_x x$ then $\tau(a) = a_1$. Since $a_x = \tau(x^{-1} x)$, we have $a = \sum_{x \in W} \tau(x^{-1} x) x$.

Let $U \subseteq V$ be $F'[W]$-modules and let $\sigma: U \to F'[W]$ be a given $F'[W]$-homomorphism. Then $\tau \sigma: U \to F$ and since $F$ is a field we can extend $\tau \sigma$ to a map $\varphi: V \to F$ which is $F$-linear.

We now define $\tilde{\varphi}: V \to F'[W]$ by
\[ \tilde{\varphi}(v) = \sum_{x \in W} \varphi(v x^{-1}) x. \]

Then certainly $\tilde{\varphi}$ is an $F$-linear map. Suppose $x, g \in W$. Then
\[ \tilde{x} g = a \tilde{x} g \quad \text{for some } a \in F, a \neq 0 \]
so we have
\[ g^{-1} \tilde{x}^{-1} = (\tilde{x} g)^{-1} = (a \tilde{x} g)^{-1} = \tilde{x} g^{-1} a^{-1} \]
and since $\varphi$ is $F$-linear
\[ \varphi(v g^{-1} x^{-1}) \tilde{x} g = \varphi(v \tilde{x} g^{-1} a^{-1}) a \tilde{x} g = \varphi(v \tilde{x} g^{-1}) \tilde{x} g. \]

Now let $g \in W$. Then by the above
\[ \tilde{\varphi}(v g^{-1}) = \left( \sum_{x \in W} \varphi(v g^{-1} x^{-1}) \tilde{x} g \right) g^{-1} = \left( \sum_{x \in W} \varphi(v \tilde{x} g^{-1}) \tilde{x} g \right) g^{-1} = \tilde{\varphi}(v) g^{-1}. \]

Thus $\tilde{\varphi}$ is an $F'[W]$-homomorphism.
Finally for $u \in U$,
\[
\tilde{\phi}(u) = \sum_{x \in W} \phi(u x^{-1}) x = \sum_{x \in W} \tau(\sigma(u x^{-1})) x
\]
\[
= \sum_{x \in W} \tau(\sigma(u)) x^{-1} x = \sigma(u)
\]
since $\sigma$ is an $F^t[W]$-homomorphism. Thus $\tilde{\phi}$ extends $\sigma$ and $F^t[W]$ is injective.

**Lemma 5.** Let $a \in K[G]$. If $a$ is not a left divisor of zero, then there exists $\gamma \in K[G]$ such that $\theta(\gamma x)\gamma = \gamma x a \gamma$ is central and is not a zero divisor in $K[G]$.

**Proof.** Write $x = \sum_{i} x_{i} z_{i}$ with $z_{i} \in K[\Delta]$ and with the $x_{i}$ in distinct cosets of $\Delta$. Let $H$ be the subgroup of $G$ generated by the elements in the support of all $z_{i}$ and their finitely many conjugates. Then $H$ is a finitely generated normal subgroup of $G$ and $H \subseteq \Delta(G)$. We use the results and notation of Lemma 1.

Now $x_{i} \in K[H] \subseteq E = K[Z]^{-1} K[H]$ so we can define $I = E x_{1} + E x_{2} + \cdots + E x_{n}$ to be a finitely generated left ideal of $E$. Observe that $E \cong F^t[H/Z]$; so by Lemma 4, $E$ is self-injective. Thus by Lemma 3 either $I = E$ or $I$ has a proper right annihilator.

Suppose $I$ has a proper right annihilator element, say $\eta^{-1} \delta \in E$ with $\eta \in K[Z]$, $\eta \neq 0$, $\delta \in K[H]$, $\delta \neq 0$. Then clearly $I \delta = 0$ so $\alpha \delta = 0$ for all $i$ and hence $\alpha \delta = 0$. Since $\alpha$ is not a left divisor of zero this is a contradiction and we conclude that $I = E$.

Now $I = E$ so $1 \in I$. If we write 1 as a sum of left $E$ multiples of the $x_{i}$ and then rationalize the denominators we see that there exists $\beta \in K[Z]$, $\beta \neq 0$, with $\beta \in K[H] x_{1} + K[H] x_{2} + \cdots + K[H] x_{n}$. Now the finite group $\Gamma C_{G}(H)$ acts on $K[Z]$ and let $y_{1}, y_{2}, \cdots, y_{n} = 1$ be a full set of coset representatives for $C_{G}(H)$ in $G$. Set $\beta = \sum \gamma_{i} x_{i}$ and for all $\gamma \in \Gamma C_{G}(H)$ we see that $\gamma \beta \neq 0$ and in fact $\gamma \beta$ is a zero divisor in $K[G]$. Moreover since $Z$ is abelian, all $\gamma x_{i}$ commute and hence $\gamma \beta$ is clearly central in $K[G]$.

Since $y_{1} = 1$, $\beta \in \sum K[H] x_{i}$, so we can write $\beta = \sum \gamma_{i} x_{i}$ with $\gamma_{i} \in K[H]$. Set $\gamma = \sum \gamma_{i} x_{i}^{-1} x_{i}$. Then $\gamma x_{i} = \sum_{i} \gamma_{i} x_{i}^{-1} x_{i} x_{i} x_{i}$. If $i \neq j$, then clearly $\text{Supp}(\gamma x_{i}^{-1} x_{i} x_{j})$ is disjoint from $\Delta(G)$. Thus $\theta(\gamma x_{i}) = \sum \gamma_{i} x_{i} = \beta$ and the lemma is proved.

**Proof of the Theorem.** Clearly $Z(Q) \cong Z(K[G])$. Let $\gamma \in Z(K[G])$ be an element which is not a zero divisor in $Z(K[G])$. Then by Lemma 2, $\gamma$ is not a zero divisor in $K[G]$ and $\gamma$ is invertible in $K[G]$. Clearly $\alpha^{-1} \in Z(Q)$.

Now let $\rho \in Z(Q)$. Then $\rho = \alpha^{-1} \beta$ where $\alpha, \beta \in K[G]$ and $\alpha$ is not a zero divisor in $K[G]$. Thus for all $w \in Q$ we have $\omega \alpha^{-1} \beta = \alpha^{-1} \beta \omega$ so $\alpha \omega \alpha^{-1} \beta \omega = \beta \omega$. Set $\omega = x_{2} x_{1}$. Then for all $x \in G$, $x x_{2} = \beta x_{2}$. Now by Lemma 5 there exists $\gamma \in K[G]$ such that $\theta(\gamma x) \in Z(K[G])$ is not a zero divisor in $K[G]$. Multiplying
the above equation on the left by $\gamma$ then yields $(\gamma x)x\beta = (\gamma \beta)x\alpha$ for all $x \in G$ and by Lemma 1.3 of [1] we have $\theta(\gamma x)\beta = \theta(\gamma \beta)x$.

Set $\delta = \theta(\gamma x)$, $\eta = \theta(\gamma \beta)$. Since $\delta \in \mathbb{Z}(K[G])$ is not a zero divisor in $K[G]$ we have $\delta^{-1}\eta \in \mathbb{Q}$ and since $\alpha^{-1}\beta \in \mathbb{Q}$ we obtain, from $\delta \beta = \eta \alpha$,

$$\delta^{-1}\eta = \beta^{-1} = \alpha(\alpha^{-1}\beta)x^{-1} = \alpha^{-1}\beta = \rho.$$

Finally $\delta$, $\rho \in \mathbb{Z}(Q)$ so $\eta = \delta \rho \in \mathbb{Z}(Q) \cap K[G] = \mathbb{Z}(K[G])$ and $\rho = \delta^{-1}\eta$ is a quotient of elements in $\mathbb{Z}(K[G])$. The result follows.

REFERENCES


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