A NONNORMAL OUTER FUNCTION IN $H^p$

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Abstract. In this paper we construct an outer function in $H^p$ for all $p$ which is not normal.

The purpose of this note is to answer a question raised by Professor Joseph Cima; namely, we show the existence of nonnormal outer functions which are in $H^p$. See [1] for relevant definitions.

The existence of a function in $H^p$ for all $p$ which is not normal easily follows from a theorem of P. Lappan [2, p. 190] which says that given an unbounded function $f$, holomorphic on the unit disc $D$, then there exists a Blaschke product $B$ such that $Bf$ is not normal. However $Bf$ is clearly not outer.

We shall construct an outer function with two different asymptotic values at $z=1$, which will imply by a theorem of O. Lehto and K. I. Virtanen that this function is not normal.

THEOREM [3, THEOREM 2, p. 53]. Let $f$ be meromorphic and normal in $G$, and let $f$ have an asymptotic value $\alpha$ at a boundary point $P$ along a Jordan curve lying in the closure of $G$. Then $f$ possesses the angular limit $\alpha$ at the point $P$.

We present the example.

$$f(z) = \frac{1}{z} \log \left( \frac{1+z}{1-z} \right) \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log t \, dt \right) = \frac{1}{z} F_1 F_2.$$  

Proof. $f \in H^p$ for all $p$ since $F_2$ is bounded and $F_1$ is in $H^p$ for all $p$. From the representation theorem of outer functions, we have that $F_2$ is outer. Furthermore, $F_1$ is a schlicht function and by Theorem 3.17 in [1, p. 51] the singular part of $F_1$ is identically one. Thus $F_1/z$ is an outer function and hence $f$ is an outer function. Finally, it is an elementary calculation to show that

$$\lim_{\theta \to 0^+} |f(e^{i\theta})| = 0 \quad \text{and} \quad \lim_{\theta \to 2\pi^-} |f(e^{i\theta})| = \infty.$$

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Thus by the above stated theorem, $f$ is not normal and has all the desired properties.

An examination of the above example yields the following result:

**Lemma.** Given $\psi : [0, 2\pi] \to \mathbb{R}^+$ with

(a) $\psi \in L^p$,
(b) $\log \psi \in L^1$,
(c) $\lim_{t \to 0^+} \psi(t) = 0$,
(d) $\lim_{t \to 2\pi^-} \psi(t) = +\infty$,

then

$$\exp \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{e^{it} + z}{e^{it} - z} \right) \log \psi(t) \, dt \right)$$

is an outer function in $H^p$ which is not normal.

**References**


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